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e-Proofs: Design of a Resource to Support Proof Comprehension in Mathematics

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Abstract

This paper presents a theoretical basis for the design of e-Proofs, electronic resources to support proof comprehension in undergraduate mathematics. To begin, we frame the problems of teaching for proof comprehension, giving research background and an argument about what teacher-centred lecturing does not, and cannot, do to address these. We then describe e-Proofs, discuss the way in which they have been used in an Analysis course, and review their limitations and affordances as part of an overall educational experience. Finally, we briefly describe the development of a web-based tool for constructing e-Proofs, ways in which this tool will be used to different pedagogical ends, and associated research activity.

Introduction

Proofs in undergraduate mathematics

In many undergraduate mathematics lectures, the lecturer spends a large proportion of the time presenting proofs of theorems (Weber, 2004). Much of the prose in textbooks also consists of proofs (Raman, 2004), and there is a clear assumption that students will learn a great deal of mathematics by reading the proofs of others (Selden & Selden, 1995). This paper is about what this entails and the design of an electronic resource to support it. This introduction presents an example of the type of proof students encounter and gives some first observations about its structure and about the thinking required to understand it.

Students in a proof-based lecture course would typically be presented with proofs like that for Rolle's Theorem as shown in Figure 1. Most first courses in Analysis would include this theorem and a version of the proof, which is not atypical of proofs at this level; some would be shorter but some would be longer and more complicated, and such a course might involve around 20 such proofs. Students might also be shown an accompanying diagram; one for Rolle's Theorem is shown in Figure 2.

For many people the diagram will confirm intuitively that the theorem is correct. The proof is nonetheless provided, with the expectation that the student will attempt to understand it. This expectation is probably different from earlier mathematics courses in which the student may have been asked to study and apply *theorems* (eg. (Hughes-Hallett, Gleason et. al. [1994] introduce the Mean Value Theorem without proof and use it in exercises on Taylor polynomial approximations). It is, however, consistent with the aim that students should come to understand mathematical theories as systems of interconnected results, all proved on the basis of agreed definitions and forms of reasoning (cf. Bell, 1976; de Villiers, 1990).

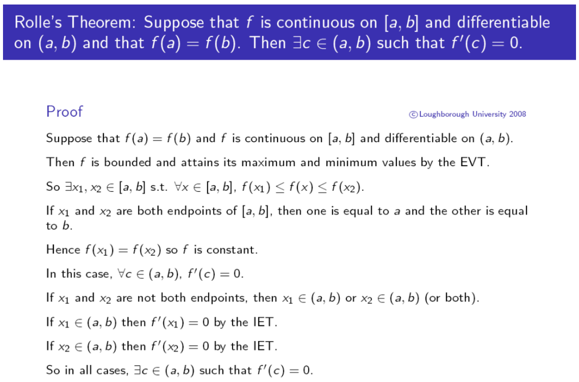

Figure 1: A proof of Rolle's Theorem – here, EVT stands for Extreme Value Theorem and IET stands for Interior Extremum Theorem

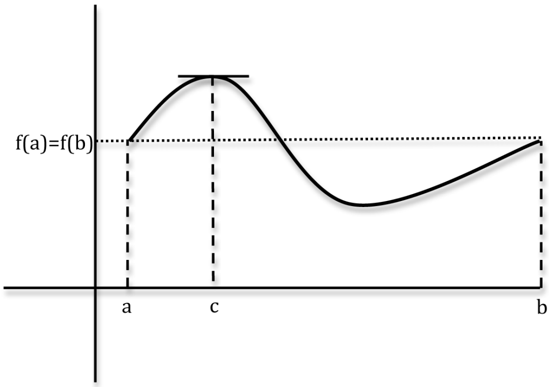

Figure 2: A diagram illustrating Rolle's Theorem

Understanding a proof: Some first observations

This proof is typical in that both the theorem and the proof are written using a combination of words and algebraic notation, the latter of which can be read out loud so that the whole proof consists of well-formed English sentences. Clearly, a student will need to be familiar with the names and meanings all of these words and symbols in order to read the proof fluently and understand it (notation used is summarised in Appendix A).

The proof is structured so that it begins with the assumptions from the theorem premises and ends with the conclusion. The first three lines prove that there exists a point x_1 at which f has a minimum on the interval – all function values on the interval are greater than or equal to $f(x_1)$ – and a point at which it has a maximum. The remaining lines rely on this information, and collectively form a subproof by cases that, whether or not this maximum and minimum occur at the endpoints, there is necessarily a point at which the derivative of the function is zero. For a full understanding, the reader will need to recognise this structure.

The proof explicitly quotes two other theorems, the Extreme Value Theorem and the Interior Extremum Theorem, both of which would probably have been proved earlier in the same course and are used without being written out. The proof also uses various defined concepts. Again, these definitions are not written out in the proof, and unlike the theorems, they are not explicitly invoked. Indeed, some (maximum and minimum) are used directly, but some (continuity and differentiability) are built into the assumptions of the quoted theorems. For a full understanding, the reader will therefore need to recall these definitions and theorems and examine the ways in which they are being used (all of these definitions and theorems are listed in Appendix A).

Clearly it is not a trivial exercise to identify this structure or to recall and examine the relevant information. In the next section, we give a more detailed theoretical breakdown of the skills this requires and describe research indicating the degree to which we can expect students at the undergraduate level to have these skills.

Theoretical issues: Framing the problem

Literature on students and proof

The literature on students' experience of mathematical proof has documented difficulties in constructing proofs (eg., Harel & Sowder, 1998; Moore, 1994; Weber, 2001) and in validating proofs or assessing whether types of argument are acceptable (eg. Healy & Hoyles, 2000; Knuth, 2002; Raman, 2003; Recio & Godino, 2001; Segal, 2000; Selden & Selden, 2003). This work is relevant to proof comprehension as discussed below, although various authors have noted that the issue of *reading* proofs has received comparatively little research attention (Hazzan & Zazkis, 2003; Mamona-Downs & Downs, 2005; Selden & Selden, 2003).

Approaches to proof comprehension

Not everyone goes about proof comprehension in the same way, and one approach is to examine how the statements and arguments relate to particular examples or diagrams. Weber (2008), for instance, documented cases in which mathematicians used examples while validating proofs, and Movshovitz-Hadar and Hazzan (2004) reported on a lecturer who used an extended example to motivate and explain a theorem and proof in group theory. Such a tactic might be called a semantic approach, by analogy with semantic proof construction strategies described in Weber and Alcock (2004) and Alcock and Inglis (2008). e-Proofs, however, focus on supporting comprehension by explicating the relationships among the theorem premises and conclusions, the individual lines of the proof, and external information such as established definitions and theorems. This might be considered a syntactic approach, again by analogy with a proof construction strategy in which the reasoner proceeds "by moving between agreed configurations such as definitions and theorems statements by applying the rules of logic, standard proof frameworks and so on" (Alcock & Inglis, *ibid.* p.115).

In the next sections we give a theoretical breakdown of a syntactic approach to proof comprehension, organising this discussion around Lin and Yang's characterisation of facets of proof comprehension. Lin and Yang identified these facets on the basis of existing literature and interviews with mathematicians and mathematics teachers (Yang & Lin, 2008). They used them to design proof comprehension questions for a purportedly student-produced proof in geometry, and used the resulting test as part of an empirical study (Lin & Yang, 2007). For each facet we do the following: 1) describe its meaning and its operationalisation via comprehension test questions; 2) compare with observations about the Rolle's Theorem proof from the introduction and with proof comprehension questions from Conradie and Frith (2000) based on a standard proof that is irrational (reproduced in the Appendix B); and 3) discuss what other research literature tells us about relevant student competencies.

Basic knowledge

Lin and Yang's first facet is called *basic knowledge*, which they operationalised as *recognising the meaning of a symbols in a figure* and *explaining/recognising the meaning of a property* (Lin & Yang, 2007 p.750). They tested basic knowledge via questions that asked for labelling figures, comparing angles etc. Conradie and Frith (2000, p.227) included comparable questions on background conceptual or procedural knowledge, for instance requesting definitions:

- How is $\sqrt{20}$ defined?
- When is a real number irrational?

As noted in the introduction, basic knowledge of definitions and earlier theorems would also be necessary to understand the proof of Rolle's Theorem. In addition, students some way into an undergraduate degree would be expected to be fluent in various forms of algebraic manipulation, to be able to state the meaning of symbols like "∃" and to be able to correctly interpret sentences containing these.

Unfortunately, we cannot expect that students will necessarily have the required background knowledge. New definitions and theorems appear on a daily basis in undergraduate courses, and it is unrealistic to think that students will have all of these at their fingertips. Also, research indicates that undergraduates are often inaccurate in interpreting the logic of mathematical statements involving conditionals and quantifiers (Dubinsky & Yiparaki, 2000; Epp, 2003; Hazzan & Leron, 1996; Selden & Selden, 1995). Further, students often do not attend to definitions, instead relying on concept images (Vinner, 1991), even when working with concepts for which they have minimal prior experience (Edwards & Ward, 2004). This is important because of the way in which precise statements of definitions are used in proofs. For instance, in the Rolle's Theorem proof, the formal definitions of minimum and maximum are combined and used to formulate line 3; a student who has only an intuitive idea that the maximum occurs "where the function is biggest" is not likely to recognise this. Of course, seeing mathematical language and definitions used in proofs is one way in which students learn about them, but inexperience will impede comprehension of any given proof in the meantime.

Logical status (inferring warrants)

Lin and Yang's second facet is called *logical status*, which they operationalised as *recognising a condition applied directly*, *judging the logical order of statements* and *recognising which properties are applied* (Lin & Yang, 2007 p. 351). They tested this via questions about possible reordering of lines and about which properties are used at different stages. Conradie and Frith (2000, p.227) included questions with similar aims, for example:

- Why may we assume that m and n have no factors in common?
- Given that 5 is a factor of m^2 how does it follow that 5 is a factor of m?

Such questions require two things. First, the reader needs to shift their focus from the content of each statement to its status; to see statements in the proof as premises and conclusions, and indeed to be able to treat the same statement as conclusion at one stage and premise at the next (Duval, 2007). Second, the reader must *infer the warrant* that the proof's author is using in order to justify the new statement. We use this term in the sense of Weber and Alcock (2005), who use a restricted version of Toulmin's (1958) scheme in which an argument is seen as composed of data, warrant and conclusion. For instance, in the second of Conradie and Frith's questions, the data is that 5 is a factor of m^2 and the conclusion is that 5 is a factor of m. Both of these appear in the proof. The question asks the reader to infer the warrant, which does not. Weber and Alcock point out that this is common: readers often have to infer warrants because these are often implicit in textbook proofs. In fact, when inferring warrants, the focus might need to be broadened to other lines of the proof, because it is common for the data to be distributed across the preceding lines and the theorem premises. For instance, in the Rolle's Theorem proof, use of the Interior Extremum Theorem requires the function to have a maximum or minimum on the interior of an interval, as assumed in line 7, and requires the function to be differentiable on that interval, as assumed in line 1.

Again, it is not realistic to assume that students will be able to do all of this easily. First, students will be accustomed to everyday argumentation in which the focus is on the content of the statements rather than on their status within a larger structure (Duval, 2007). Second, a student who does not accurately interpret conditional and quantified mathematical statements is unlikely to infer warrants appropriately, and a student who is not conversant with earlier definitions and theorems will be further hampered in this process. Third, and more importantly, research on proof validation indicates that students *may not even attempt* to infer warrants when reading proofs. Selden & Selden (2003), for instance, found that students who were asked to check the validity of short number theory proofs often did not notice when one line did not follow from the line above. Similarly, Alcock and Weber (2005) found that only two out of 13 undergraduate students correctly inferred and rejected a fairly straightforward invalid warrant in an Analysis proof. Weber (2009) found that 28 undergraduates who had completed a transition-to-proof course rarely spent more than two minutes deciding whether purported proofs were valid. They were often prepared to make a validity judgment despite acknowledging their own incomplete understanding; at least some appeared to believe it to be the responsibility of the proof's author to spell out all the details, so that the fault in understanding in these cases lay with the author and not with the reader. This indicates that a substantial number of students may not read proofs in a way that is likely to lead to understanding of their logic.

Summarisation (identifying larger scale structure)

Lin and Yang's third facet is *summary*, which they operationalised as *identifying critical procedures, premises or conclusions* and *indentifying critical ideas of a proof* (Lin & Yang, 2007, p.751). They tested this via questions about what the proof shows and about identifying a significant intermediate result and how it is used. Conradie and Frith (2000, p.227-228) again included questions with similar aims, such as:

- What method of proof is used here?
- Which assumption is contradicted, and how does the theorem follow from this?

The introduction to this paper discussed such overall structure for the proof of Rolle's Theorem. Duval (2007, p.142) captured another such structure with the aid of a tree diagram for a geometry proof in which the theorem premises are used to prove two independent intermediate results, which are then put together to arrive at the required conclusion.

Identifying such structure requires understanding the proof at a more global level, looking for major steps, subproofs and standard structures within these subproofs or the proof as a whole. Again, accurate interpretation of conditional and quantified statements will be required to do this fully. Knowledge of definitions will be highly relevant because statements like "Prove that *x* is an *X*" need to be interpreted as "Prove that *x* satisfies the definition of *X*"(cf. Alcock and Simpson, 2002), meaning that definitions often form structures for proofs. Selden and Selden (2003) discussed this point in detail, arguing that definitions and other statements often dictate the top-level structure of a proof. Difficulty in identifying larger-scale structures is likely to be exacerbated by confusion over particular argument structures such as proof by induction (eg., Dubinsky, 1987; Harel, 2001) or contradiction (eg., Antonini & Mariotti, 2008).

Generality

Lin and Yang's fourth facet is *generality*, which they operationalised as *justifying correctness* and *identifying what is validated by the proof* (Lin & Yang, 2007, p.751)[1]. They tested this via questions that asked whether the purported proof was valid and whether it proved that the target statement was sometimes or always correct. Conradie and Frith included a question in which a proof appeared without its corresponding theorem and the student was asked what had been proved (Conradie & Frith, 2000, p.228).

Questions of this type might seem irrelevant to a lecture in which a correct theorem and proof is presented, so that there is no question of validity or scope. However, we would still want the student to understand that the proof does in fact prove the specified statement, and this might be problematic. Selden and Selden (2003) found that in validating short purported number theory proofs, only two out of eight students initially spotted that one of these was a proof of the converse of the target theorem (also incorporating a notational error). Weber (2009) reported similar results. It is worth noting that in these studies, along with that of Alcock and Weber (2005), many of the arguments used were only four lines long. This is substantially less than the length of many proofs presented in undergraduate lectures.

Combining skills

Describing these facets in this order emphasizes that some are more local (understanding particular lines) and others more global (understanding overall structures). This should not be taken to mean that one proceeds in understanding a proof in this order. Weber (2008), for instance, found that mathematicians faced with proofs in an unfamiliar area typically began by identifying the global structure and then proceeded to a line-by-line check. Clearly, however, developing a full understanding of a proof is a complicated process, and there is a lot of scope for proofs to be poorly understood.

Practical issues: The problem of lecturing

Research-based responses to difficulties with proof

In recognition of students' difficulties, mathematics educators have suggested various ways of making proofs more accessible. Rowland (2001), for instance, suggested that proofs using a generic example might be more comprehensible than fully general proofs. Harel (2001) described an approach to proof by induction that begins with repeated experience of constructing recursion arguments. Leron (1985), described both an approach to contradiction that involves working first on the central constructive idea, and a general approach in which a proof task is broken into chunks to highlight its overall structure (Leron, 1983). Others have focused on student-centered reform-oriented instruction in which entire courses have been redesigned in order to give students more responsibility for constructing proofs (eg., Alcock & Simpson, 2001; Rasmussen and Marrongelle, 2006; Zandieh, Larsen and Nunley, 2008).

However, neither type of research has not had a strong influence on how proofs are presented. In the latter case this is partly because of resource issues: many lecture classes involve well over 100 students, a situation that does not lend itself to involving the students as a coherent knowledge-building community and that is not going to change any time soon. While large lectures do not have to be run entirely on a transmissionist model (eg., Biggs & Tang, 2007), lecturer-provided explanations are likely to remain a mainstay of the undergraduate mathematical experience for the foreseeable future.

Explanations in lectures

In presenting a proof, a lecturer typically writes it on the board one line at a time, giving additional explanation about why each line is valid. He or she might also give an overview of the argument, state rationales for certain approaches, point out sections that achieve different subgoals, and relate these to the overall structure of the theorem (eg., Movshovitz-Hadar & Hazzan, 2004; Weber, 2004).

These explanations may be clear and informative, but there are several problems with expecting them to lead to proof comprehension. First, they require the student to draw on background knowledge, recognise and validate cited warrants, and recognise larger scale structures and generality, all in rapid succession. Second, although the lecturer will try to facilitate this process with reminders, hand gestures and so on, the student's attention may not be directed precisely enough. Third, even if a student's attention is in the right place(s), they may not be able to grasp the logical relationships quickly enough to understand them, especially if this involves recalling an earlier theorem or results from earlier lines. Fourth, each student is likely to have slightly different difficulties in following the explanation, and the lecturer cannot take a few minutes to pause for each of these. Fifth, whatever explanation is offered is ephemeral and is typically no longer available when the student comes to re-read their lecture notes. This means that even a dedicated student who pays attention in lectures must reconstruct it during independent study.

Once these problems are recognized, one practical solution would be to record the lecture. This would allow a student to see and hear explanations again, but does not address the problems of directing attention precisely or of seeing relationships in real time. Also, there may be slips and hesitations in the spoken explanation, visuals and audio are unlikely to be optimally clear, and there may be extraneous distracters in either. Another solution would be to provide additional written information to accompany the proof. This is sometimes done, perhaps in a two-column format as by the professor studied by Weber (2004). However, giving more detail might obscure the structure of the proof, as other authors have noted: "to make a proof too detailed would be more damaging to its readability than to make it too brief" (Davis & Hersh, 1985, p.73) and "[the student may] have difficulty distinguishing supplementary and explanatory remarks from the proof itself" (Selden and Selden, 1995, p. 140). Annotations and further explanation might well be useful, but we suggest that adding these as additional text is not an optimal delivery method, and that a technological solution can do better.

e-Proofs

Initial design

e-Proofs are designed to address theoretical proof comprehension issues within the practical context of traditional lectures, by making the structure and reasoning used in a proof more explicit without cluttering its presentation. Each e-Proof consists of a sequence of screens such as that shown in Figure 3. Each screen shows the theorem and the whole proof, with much of the latter "greyed out" to focus attention on particular lines. Relationships are highlighted using boxes and arrows, and each screen is accompanied by an audio file which students can listen to as many times as they wish.

The screen in Figure 3 comes from what we have termed the *line-by-line* version of this e-Proof. We also constructed *chunk* versions, the aim of which is to focus attention on the global structure of the proof by breaking it into relatively self-contained sections or subproofs. Figure 4 shows a screen from the chunk version of the same e-Proof.

Improvements in a new version

The e-Proof screens shown above were constructed by using Beamer to convert a LaTeX file into a pdf presentation, which was then annotated and separated into screens. The audio was recorded using Audacity. This content was then uploaded to the university's virtual learning environment (VLE), making use of one of its standard lesson structures. This was a somewhat clumsy process involving uploading screens and audio separately, and was restricted by the content and structure of the rest of the VLE's standard layout.

Figure 5 shows a prototype improved version made[2] using Flash. In this version, annotations are better synchronized with the audio content, so that the arrows and boxes appear and disappear exactly when they are needed. (If you are reading this article online, you can run the complete e-Proof).

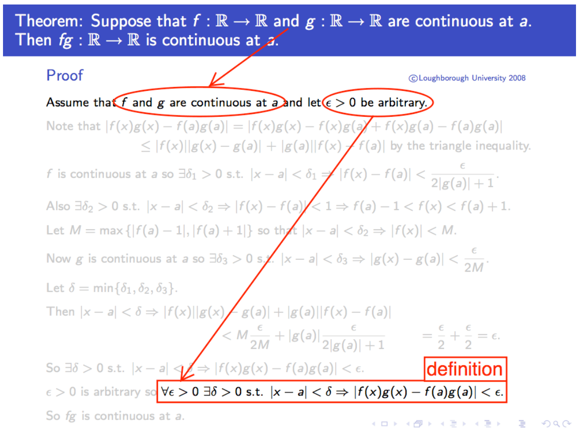

Figure 3: A screen shot from an e-Proof for the product rule for continuous functions

The accompanying audio says: "In the first line, we state our assumption that *f* and *g* are continuous at *a*, which corresponds to the premise of our theorem. We also let epsilon greater than zero be arbitrary, because we want to show that *fg* satisfies the definition of continuity at *a*, which we will achieve by the end of the proof. Doing so involves showing that something is true for all epsilon greater than zero, so choosing an arbitrary epsilon means that all our reasoning from now on will apply to any appropriate value."

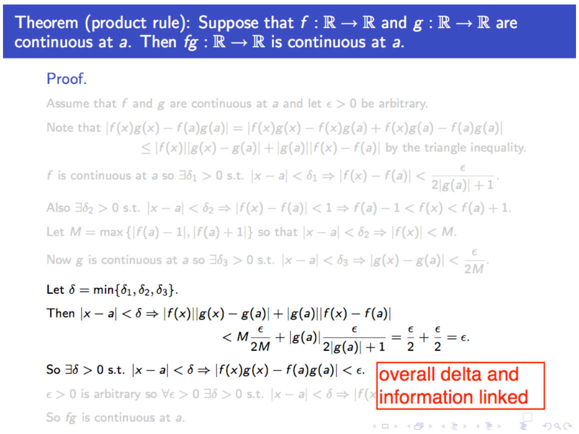

Figure 4: A screen shot from a *chunk* version of an e-Proof for the product rule for continuous functions

The accompanying audio says: "In the third chunk, we set up an overall delta value, and put together the information from the second chunk to show that if the modulus of x minus a is less than this delta, then our original modulus expression is less than epsilon."

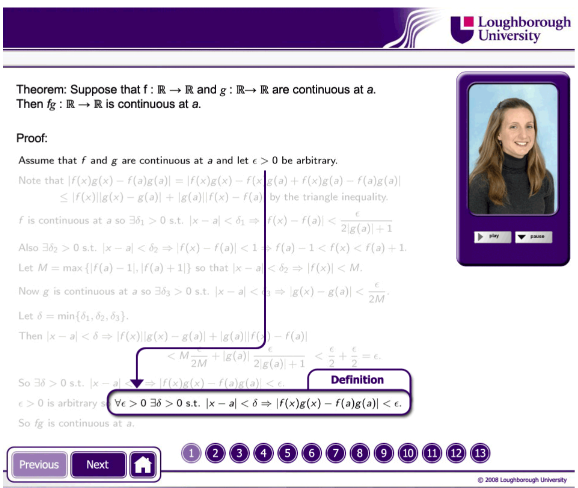

Figure 5: An improved Flash version of an e-Proof

Addressing theoretical and practical issues

In the theoretical framework section we discussed four facets from Lin and Yang's (2007) breakdown of proof comprehension: basic knowledge (relevant background procedural and conceptual knowledge), logical status (inferring warrants), summary (identifying critical ideas and subproofs) and generality (identifying what is proved). Each of these can be supported by e-Proofs.

Basic knowledge can be supported in a low-level way simply by providing correct and fluent reading of all of the words and symbols in the proof. Indeed, such reading might highlight important conceptual information, if for example a statement like "$|x - a| < \delta$" is read out loud as "the distance between x and a is less than delta". Basic knowledge can also be supported by providing audio reminders of relevant definition and theorem statements. Reading for logical status can be supported by giving explicit audio explanations of implicit warrants. Annotations can highlight which information is being used as data for a particular claim, even when this is dispersed across the proof, and can visually link this information to the conclusion. Reading in order to identify critical ideas and subproofs can be supported either by indicating their beginnings, ends and internal structure with line-by-line annotations or by breaking proofs down as in the chunk version and providing commentary on what is achieved in each section. Finally, reading in order to identify what is proved can be supported by providing a screen with arrows indicating where the theorem premises are used and where the conclusion appears.

All of this information could be provided in a lecture, but here the explanation is not only captured but enhanced by directing attention precisely and having clear visuals and audio. Low-level details are hidden but retrievable, navigation to a specific point of difficulty is straightforward, the audio can be replayed as many times as the student wishes and the reader can proceed at his or her own pace. Also, the annotations appear one at a time and do not permanently add content, so the integrity of the proof is preserved without clutter. Overall, the coordination of the static underlying proof and the dynamic annotations and audio mean that the thinking one needs to do to understand a proof is made explicit in a way that could not be achieved in a lecture or a book.

Design, implementation and usage

Design of individual e-Proofs

Designing an e-Proof requires considerable intellectual work because of two coordinated constraints: the screen size and the length of each audio explanation. Making a proof fit on one screen often requires compression compared with what might be written on a board. Such compression is possible because some of the explanation that might ordinarily appear in a board version can be put in the audio commentary ("by line 3", "this contradicts our assumption at *", and so on). On the other hand, the logic of the written version needs to remain clear, and the audio itself is also constrained. Laurillard (2002, p.110) states: "If a hyperlinked clip lasts longer than thirty seconds there is a sense of the user having ceded control, and they revert to being the viewer, rather than active participant.... Ten to twenty seconds is more comfortable." Coordinating these aspects, however, is easier than deciding on the content of the explanation and how this will relate to what is fully visible on the screen and what annotations should appear. The difficulty of constructing satisfactorily short, fixed explanations further convinced us of the likely inadequacy of the on-the-fly explanations typically given in lectures.

Implementation

The first author constructed eight e-Proofs for an Analysis course [3] that covered standard content on continuity, differentiability and Riemann integrability and was given to a cohort of 140 students in Autumn, 2008. For each e-Proof she first gave out printed copies of the theorem and proof and invited the students to spend a few minutes reading and discussing these. She then showed the line-by-line and chunk versions of the e-Proof, playing most of the audio but sometimes inviting the students to confirm that they could see how a simple line worked without it. If appropriate, she also drew a diagram on the board as the proof progressed. This whole process typically took approximately 15-25 minutes, and the students did not receive any particular instruction on what they should do while the e-Proof was shown. Subsequently she saw a number of printed copies with copious annotations, but does not know whether these were made during lecture time. After the lecture, the e-Proof was made available via the course VLE page.

This experience of using the e-Proofs led her to two main observations. First, when the first e-Proof was shown, the students seemed somewhat daunted. We believe this indicates that the e-Proof made clear how much work that might go into understanding a proof. Second, the use of e-Proofs had a noticeable effect upon her lecturing, in that she made many more comments than usual about the process of understanding proofs. She commented on what to look for in seeking line-by-line links and overall structure, related this to the experience of watching e-Proofs and indicated that this is something a student should do for every proof. In doing so she stressed that this process should take some time, but not an impossible amount. Using e-Proofs thus made the process of proof comprehension an overt subject of discussion in the lectures.

Usage

The VLE collects usage data for all the posted documents and other types of activity, so it is possible to ascertain how much the e-Proofs were actually used by the students. The e-Proofs were collectively viewed a total of 1026 times during the course and in the pre-examination period (seven viewings per student on average), with more viewings in this latter period. This was comparable with usage of other online resources such as solutions to not-for-credit weekly problem sheets. Feedback was positive, with the vast majority of students indicating that they would like e-Proofs for other courses. For more detail on usage and feedback see Alcock (2009).

Discussion: Pedagogical affordances and limitations

What e-Proofs do not do

e-Proofs were designed to address the problem of teaching for proof comprehension in large, teacher-centred undergraduate mathematics lectures. We have argued that in theory, they can focus attention on the thinking needed for syntactic proof comprehension by making explicit both warrants for line-by-line validity and larger-scale structure. However, it is important to recognize the limits on what such a resource can contribute to the overall learning process (the following is much influenced by Laurillard, 2002).

Essentially, an e-Proof allows the lecturer to articulate their own understanding of a proof. We have argued that it allows them to do this better than they could in a lecture or a standard written explanation, but it is still just an explanation. The lecturer can attempt to anticipate likely points of difficulty, but students have no opportunity to articulate their own conceptions and receive feedback on these. In this respect, e-Proofs are considerably less sophisticated than what Laurillard (2002, chapter 7) calls *adaptive media*. In mathematics education, one might see this by comparing with CAA (see eg. Sangwin, 2004), which can provide extrinsic feedback by responding to anticipated answers in particular ways, and with dynamic geometry software (see eg., Hadas, Hershkowitz & Schwarz, 2000), which can provide intrinsic feedback by allowing the student to immediately see the effect of their constructions and dragging actions.

e-Proofs *are* interactive, but only in the weak sense that the student controls the pace and sequence of the content and can replay parts at will. In discussing interactive media, Laurillard (2002, p.110) notes that “[w]ithout a clear personal goal, students will tend to iterate through the resource without either reflection or adaptation”. In this case, a student can sit in front of an e-Proof without thoughtfully engaging just as easily as they can sit in a lecture without thoughtfully engaging. In the Analysis course, e-Proofs were combined with other types of instruction and activity, some of which encouraged students to share their understanding with each other and then reflect upon it when the e-Proof was played or a solution was made available. As in any learning situation, consideration must be given to the student’s perception of what they are supposed to be learning and how the learning activities and resources are supposed to support that (see eg., Ramsden, 2003).

Continuing work

Research and teaching

e-Proofs are designed to support proof comprehension, and with the support of an MSOR Network[4] mini-project award, a research study has been undertaken to investigate whether they actually do. This project compared students’ comprehension of a proof after a) studying an e-Proof; b) watching a lecture; and c) reading the proof independently. Roy, Alcock & Inglis (2010) report that in this first exposure to a particular proof, the lecture led to the greatest comprehension. They discuss possible reasons for this, which are closely related to the limitations of e-Proofs as discussed above. This work links to the work being carried out by the ExPOUND Project (see below), where we will be documenting how and why students use e-Proofs, both in terms of their detailed interaction with particular proofs and as part of their overall study for a course. Finally, lecturers at Loughborough will be exploring the possibility of allowing students to construct their own e-Proofs for submission as part of an assignment in a course on Communicating Mathematics. In this way it is hoped that e-Proofs will allow students not just to better understand lectured proofs, but to demonstrate their own understanding of proofs that they have studied from other sources.

ExPOUND project

With the support of a JISC[5] Learning and Teaching Innovation Grant, work is now underway to develop an open-source web-based tool called ExPOUND (Explaining Proofs: Offering Understanding through Notated Demonstrations). The tool has been designed to allow both lecturers and students to construct e-Proofs as illustrated in the improved prototype version as shown in Figure 5. The tool itself is written in Flex and PHP, using both rapid prototyping and agile development practices, and has been released under an open source license so that it can be installed for use at other institutions and the underlying code can be modified for bespoke functionality. The individual ExPOUND user will, however, be able to construct an e-Proof through a web browser so no installation will be required. The user will be able to share their project build files so that others can make modifications for their own settings, and the finalised e-Proofs will be a Flash files, allowing easy sharing of these learning objects via, for example, an institutional VLE.

The ExPOUND team has gathered early feedback by meeting with lecturers interested in being able to use both the ExPOUND tool and the e-Proof products in their teaching practice. Early indications have been positive: the lecturers engaged with the team, noted potential limitations and suggested additional features that would be useful to them. Many of these suggestions have been incorporated into the initial tool and/or documentation. Early interest has been from mathematicians, but lecturers from other disciplines such as design and technology and chemical engineering are also beginning to express interest in using ExPOUND to construct learning objects for their own subject areas. The tool has been developed with inbuilt flexibility to allow for such cross-discipline use in future.

Work is currently underway to provide an online demonstrator that lecturers can trial; feedback received will allow the project team to add enhancements for future versions of the tool. The project also aims to make the final e-Proofs available as resources on the projects website, as they are developed. Those who are interested in following the development of the tool or engaging with the project are encouraged to follow its progress online at http://www.projectexpound.org.uk/.

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Footnotes

[1] Lin and Yang also have a final facet, *application*. This is concerned with the applicability of a theorem or proof to results other than the theorem in question, so we do not consider it here.

[2] Constructed by Lee Barnett.

[3] This work was supported by a Loughborough University Academic Practice Award and was completed with assistance from Lee Barnett and Keith Watling.

[4] Mathematics, Statistics and OR Network, see http://www.ltsn.gla.ac.uk

[5] Joint Information Systems Committee, see http://www.jisc.ac.uk

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