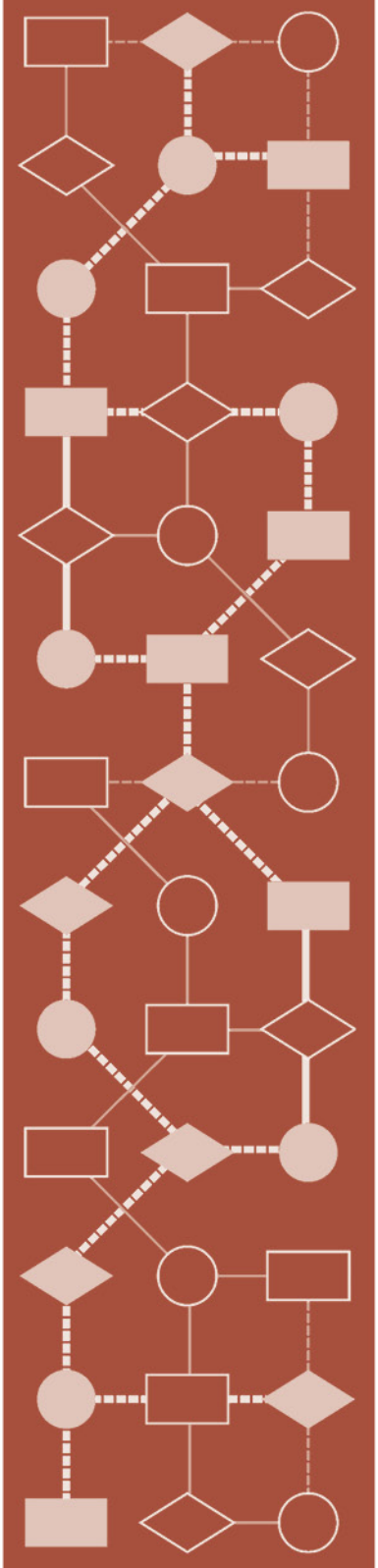
PROBLEM SOLVING



Mathematics Assessment Project

CLASSROOM CHALLENGES

A Formative Assessment Lesson

Estimating: *Counting Trees*

Mathematics Assessment Resource Service
University of Nottingham & UC Berkeley

Beta Version

Estimating: *Counting Trees*

MATHEMATICAL GOALS

This lesson unit is intended to help you assess how well students are able to:

- Solve simple problems involving ratio and direct proportion.
- Choose an appropriate sampling method.
- Collect discrete data and record them using a frequency table.

COMMON CORE STATE STANDARDS

This lesson relates to the following *Mathematical Practices* in the *Common Core State Standards for Mathematics*:

1. Make sense of problems and persevere in solving them.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.

This lesson gives students the opportunity to apply their knowledge of the following *Standards for Mathematical Content* in the *Common Core State Standards for Mathematics*:

7.RP: Analyze proportional relationships and use them to solve real-world and mathematical problems.

7.G: Solve real-life and mathematical problems involving angle measure, area, surface area, and volume.

7.SP: Use random sampling to draw inferences about a population.

INTRODUCTION

The unit is structured in the following way:

- Before the lesson, students attempt the *Counting Trees* task individually. You then assess their responses, and formulate questions that will prompt students to review their work.
- At the start of the lesson, students think individually about their responses to the questions set.
- Next, students work in small groups to combine their thinking and work together to produce a collaborative solution to the *Counting Trees* task, in the form of a poster.
- Working in the same small groups, students evaluate and comment on sample responses, identifying the strengths and mistakes in these responses and comparing them with their own work.
- In a whole-class discussion, students compare and evaluate the methods they have seen and used.
- Finally, students reflect on their work and their learning.

MATERIALS REQUIRED

- Each individual student will need a copy of the task *Counting Trees,* at least one copy of *The Tree Farm*, and a copy of the questionnaire *How Did You Work?*
- Each small group of students will need at least one enlarged copy of *The Tree Farm*, a sheet of poster paper, a glue stick, felt tipped pens, and copies of *Sample Responses to Discuss*.

There are some projector resources to help introduce activities and support whole-class discussions.

TIME NEEDED

15 minutes before the lesson, a 70-minute lesson and 10 minutes in a follow-up lesson (or for homework.) Timings given are only approximate. Exact timings will depend on the needs of your class.

BEFORE THE LESSON

Task: *Counting Trees* (15 minutes)

Have the students complete this task, in class or for homework, a few days before the formative assessment lesson. This will give you an opportunity to assess their work, and to find out the kinds of difficulties students have with it. You will then be able to target your help more effectively in the follow-up lesson.

Briefly introduce the task.

Does anyone know what a tree farm is?

How is a tree farm different from a natural forest?

[The trees are deliberately grown for commercial purposes; e.g. building or paper. They are often planted in rows.]

Give each student a copy of *Counting Trees* and *The Tree Farm.*

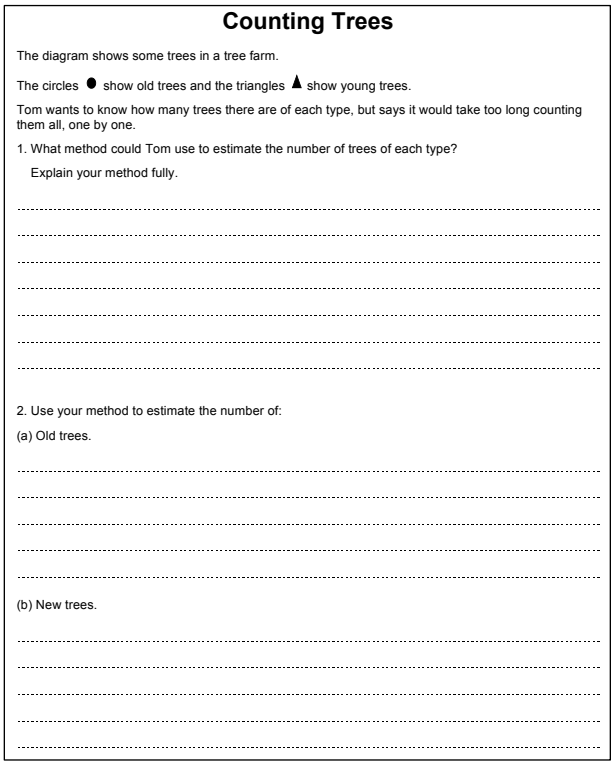
Spend fifteen minutes on your own, reading through the questions and trying to answer them as carefully as you can.

I have extra copies of the sheet, The Tree Farm, *if you need another copy.*

Don't worry if you can't do everything. There will be a lesson on this material that will help you to improve your work. Your goal is to be able to answer these questions with confidence by the end of that lesson.

You may want to show the class Slide P-1 of the projector resource.

It is important that students are allowed to answer the questions without assistance, as far as possible. If students are struggling to get started then ask questions that help them understand what is required, but make sure you do not do the task for them. You may need to remind students that Tom does not want to count all the trees in the tree farm.



Counting Trees

The diagram shows some trees in a tree farm.

The circles ● show old trees and the triangles ▲ show young trees.

Tom wants to know how many trees there are of each type, but says it would take too long counting them all, one by one.

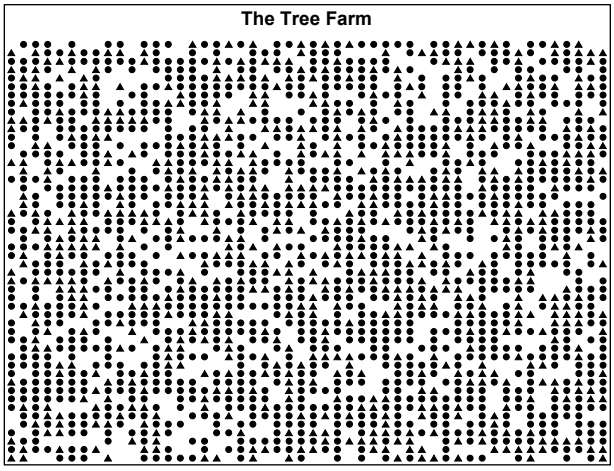
1. What method could Tom use to estimate the number of trees of each type?

Explain your method fully.

2. Use your method to estimate the number of:

(a) Old trees.

(b) New trees.



The Tree Farm

Students who sit together often produce similar answers, and then when they come to compare their work, they have little to discuss. For this reason, we suggest that when students do the task individually, you ask them to move to different seats. Then at the beginning of the formative assessment lesson, allow them to return to their usual seats. Experience has shown that this produces more profitable discussions.

When all students have made a reasonable attempt at the task, tell them that they will have time to revisit and revise their solutions later.

Assessing students' responses

Collect students' responses to the task. Make some notes on what their work reveals about their current levels of understanding, and their problem solving strategies.

We suggest that you do not score students' work. The research shows that this will be counterproductive, as it will encourage students to compare scores, and distract their attention from what they can do to improve their mathematics.

Instead, help students make further progress by summarizing their difficulties as a list of questions. Some suggestions for these are given the *Common issues* table. We suggest that you make a list of your own questions, based on your students' work, using the ideas below. We strongly recommend that you write questions on each individual student's work, but if you do not have time to do this, select a few questions that will be of help to the majority of students. These can be written on the board at the beginning of the lesson.

Common issues:	Suggested questions and prompts:
Student chooses a method which does not involve any sampling For example: The student counts the trees. Or: The student multiplies the number of columns by the number of rows, and then halves this answer.	• Read the question again. Have you done what is asked? • What assumptions have you made? Are your assumptions reasonable? • How could you improve your estimate?
Student chooses a sampling method that is unrepresentative For example: The student counts the trees in the first row/column and multiples by the number of rows/columns. Or: The student multiplies the number of trees in the left column by the number of trees in the bottom row.	• How could you improve/check your estimate? • Is your sample size reasonable? How do you know? • Which rows/columns have you left out of your calculations?
Student uses area and perimeter in their calculations	• What does the area measure? • What does the perimeter measure?
Student makes incorrect assumptions For example: The student does not account for gaps. Or: The student does not realize that there are an unequal number of trees of each kind.	• Is there a pattern to how the trees are distributed in the tree farm? Does your work assume there is a pattern? • What does your method assume? Is this a reasonable assumption?
Student calculates the number of trees in an unrepresentative sample area of the tree farm	• Is your sample area representative of the whole tree farm? • How could you check the accuracy of your estimate?
Students' work is difficult to follow	• Would someone unfamiliar with the task understand your work?
Student chooses an appropriate sampling method	• Can you suggest a second, different sampling method? • If you miscount your sample by 1, how does that affect your overall estimate?
Student completes the task	• Now have a go at this problem. How many people can stand on a full-size tennis court? State your assumptions and come up with a reasonable estimate.

SUGGESTED LESSON OUTLINE

Reviewing individual solutions to the problem (10 minutes)

Return your students' work on *Counting Trees*.

If you have not added questions to individual pieces of work, write your list of questions on the board. Students are to select questions from the board that are appropriate to their own work, and spend a few minutes thinking about their responses to them.

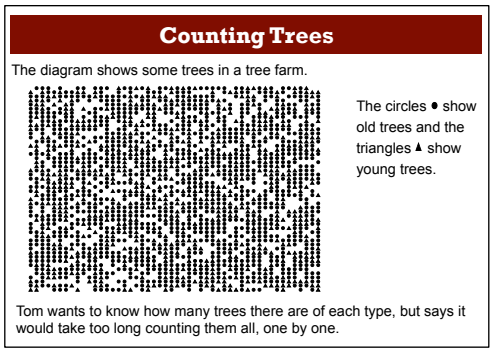
Recall what we were working on previously. What was the task about?

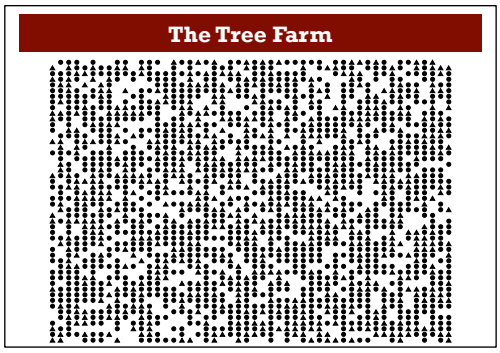
I have had a look at your work, and I have some questions I would like you to think about.

Carefully read through your own work and the questions I have written. I would like you to think, on your own, about my questions and how your work could be improved.

This is an opportunity for students to review their work. Whilst they are doing this, it may be appropriate to ask individual students questions that help them to clarify their thinking.

You may want to show the class Slide P-1 or Slide P-2 of the projector resource.





When a list of questions is written on the board rather than on individual pieces of work, some students struggle to identify which questions they should be considering. If this is the case, it may be helpful to give students a printed version of the list of questions with the relevant ones highlighted.

Collaborative work: making posters (20 minutes)

Organize the class into small groups of two or three students and hand out an enlarged copy of the sheet *The Tree Farm*, poster paper, a glue stick, and felt-tipped pens.

Ask students to have another go at the task, but this time ask them to combine their ideas and use what they have learned from reviewing their individual solutions.

You each have your own individual solution and have been thinking about how you might improve this.

Share your method with your partner(s) and your ideas for improving your work.

Together in your group, agree on the best method for completing the problem and produce a poster, which shows a joint solution to the task that is better than your individual solutions.

State on your poster any assumptions you have made and give clear reasons for your choice of method.

Slide P-3 of the projector resource summarizes these instructions.

Ask students to glue the sheet *The Tree Farm* to their poster, as part of their solution. They should show their method clearly on the tree farm diagram as well as writing an explanation on their poster.

Have extra enlarged copies of *The Tree Farm* available for students should they request them.

While students work in small groups you have two tasks, to note their different approaches to the task, and to support student reasoning.

Note different student approaches to the task

In particular, note whether students' original methods are the same or different to their joint strategy. How do they decide which method to use? What assumptions do they make? Do students choose an appropriate sampling method? Do they check their estimate by considering a different sample? What makes them decide they have looked at enough sample areas?

Support student reasoning

Try not to make suggestions that move students towards a particular approach to the task. Instead, ask questions that help students to clarify their thinking.

What assumptions have you made? Do you think they are reasonable?

Why did you select this sample to count?

Why is it helpful to count trees in more than one sample?

How do you decide how many sample areas to count? [If there is a big variation in the number of trees then students may want to count the number of trees in a third area.]

What is the difference between an estimate and a guess?

How can you check you have a good estimate?

You may want to use some of the questions in the *Common issues* table to support your own questioning. If the whole class is struggling on the same issue, write relevant questions on the board and hold an interim discussion.

Sharing different approaches (10 minutes)

Hold a whole-class discussion on the methods used to produce a group solution. Ask two groups of students to describe the method used and the ways in which this method differs to their initial individual responses. Did the students check the accuracy of their estimate? If they did, what method did they use?

When selecting students to report back to the rest of the class in this way, try to choose groups of students that have approached the task in different ways, rather than groups that have adopted a similar approach.

Collaborative analysis of *Sample Responses to Discuss* (20 minutes)

Once students have had sufficient time to discuss some different approaches, distribute to each group, copies of the *Sample Responses to Discuss*. It may not be appropriate, or there may not be enough time, for all students to analyze three sample responses. Each response highlights different misconceptions and so depending on the progress already made on the task, it may be appropriate to issue different sample responses to different groups. For example, groups that have assumed an equal number of old and new trees could be given Laura's work, while groups that have failed to check the accuracy of their estimate could be given Wayne's work.

The whole-class discussion held after the collaborative work should help to inform your decision on whether or not to be selective about which sample responses students are given.

In your groups you are now going to look at some student work on the task. Notice in what ways this work is similar to yours and in which ways it is different.

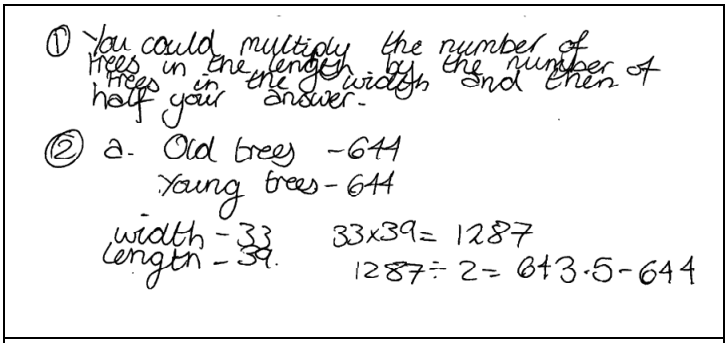
There are some questions for you to answer as you look at the work. You may want to add annotations to the work to make it easier to follow.

This task gives students an opportunity to evaluate a variety of possible approaches to the task, without providing a complete solution strategy. Students should thoughtfully answer the questions below each piece of sample student work and be encouraged to think carefully about ways in which the work could be improved.

Slide P-4 of the projector resource provides an overview of the questions the students will be answering as they analyze the sample student responses.

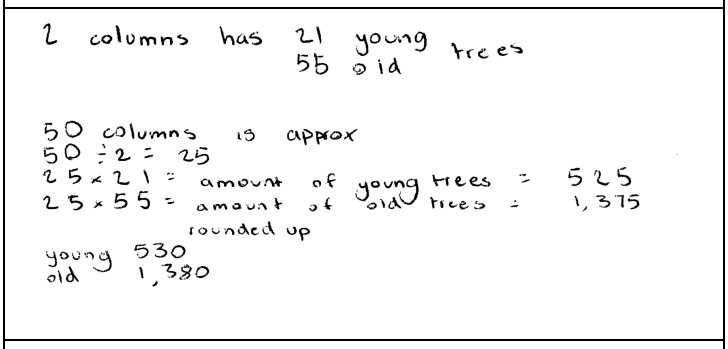
Laura attempts to estimate the number of old and new trees by multiplying the number along each side of the whole diagram and then halving. She does not account for gaps nor does she realize that there are an unequal number of trees of each kind.

Can you explain why Laura halves her answer? What assumption is she making?



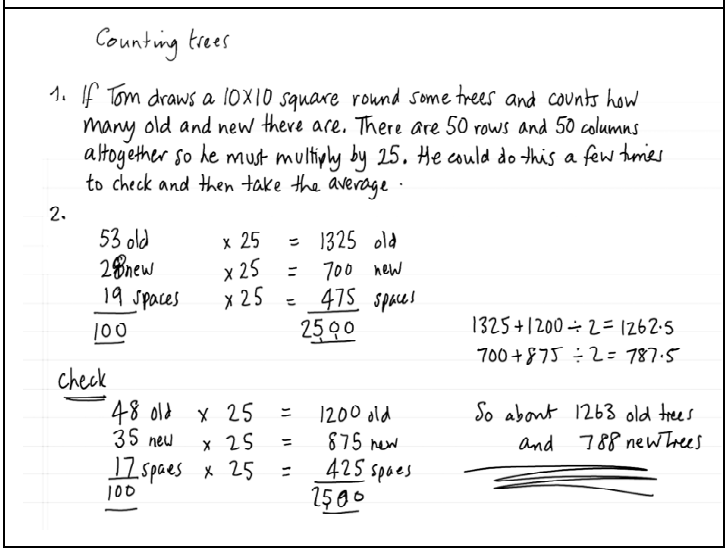
Wayne uses a sample of two columns and counts the number of old and young trees. He then multiplies by 25 (half of 50 columns) to find an estimate of the total number. He does, however, take account of the different numbers of old and new trees.

Why do you think Wayne has rounded his answers?



Amber chooses a representative sample and carries through her work to get a reasonable answer. She correctly uses proportional reasoning. She checks her work as she goes along by counting the gaps in the trees. Her work is clear and easy to follow, although a bit inefficient.

Can you explain why Amber multiplies by 25 in her method?



Whole-class discussion (15 minutes)

Discuss some of the different approaches used in the sample work and ask students to comment on their strengths and weaknesses. You may also want to compare students' own work with the sample student responses. Did any group use a similar method to Laura, Wayne or Amber? What was the same/different about the work? In what ways did analyzing the responses enable students to identify errors in their own work?

You may want to use Slides P-5, P-6 and P-7 of the projector resource and the questions in the *Common issues* table to support this discussion.

Follow-up lesson (or possible homework): individual reflection (10 minutes)

Once students have had a chance to discuss the sample responses as a whole class, distribute the questionnaire *How Did You Work?* Ask students to spend a couple of minutes, individually, answering the questions.

Think carefully about your work this lesson and the different methods you have seen and used.

This would make a suitable homework task.

SOLUTIONS

There are many ways of completing this task, but solutions should include the following:

- Students should describe an appropriate sampling technique which takes into account the different proportions of old and new trees as well as allowing for the gaps in between.
- Students should check their figures for the number of old and new trees by counting the number of trees in at least two sample areas.
- If there is a big variation in the number of trees in two areas then students should count the number of trees a third or subsequent sample area.
- Each sample should cover a distinctly different area of the tree farm (that is two sample areas should not be next to each other.)
- Each sample should cover the same sized area.
- Students may calculate the average number of old and new trees for their chosen samples. These averages should then be multiplied up to obtain an estimate for the total number of old and new trees.

Counting Trees

The diagram shows some trees in a tree farm.

The circles ● show old trees and the triangles ▲ show young trees.

Tom wants to know how many trees there are of each type, but says it would take too long counting them all, one by one.

1. What method could Tom use to estimate the number of trees of each type?

   Explain your method fully.

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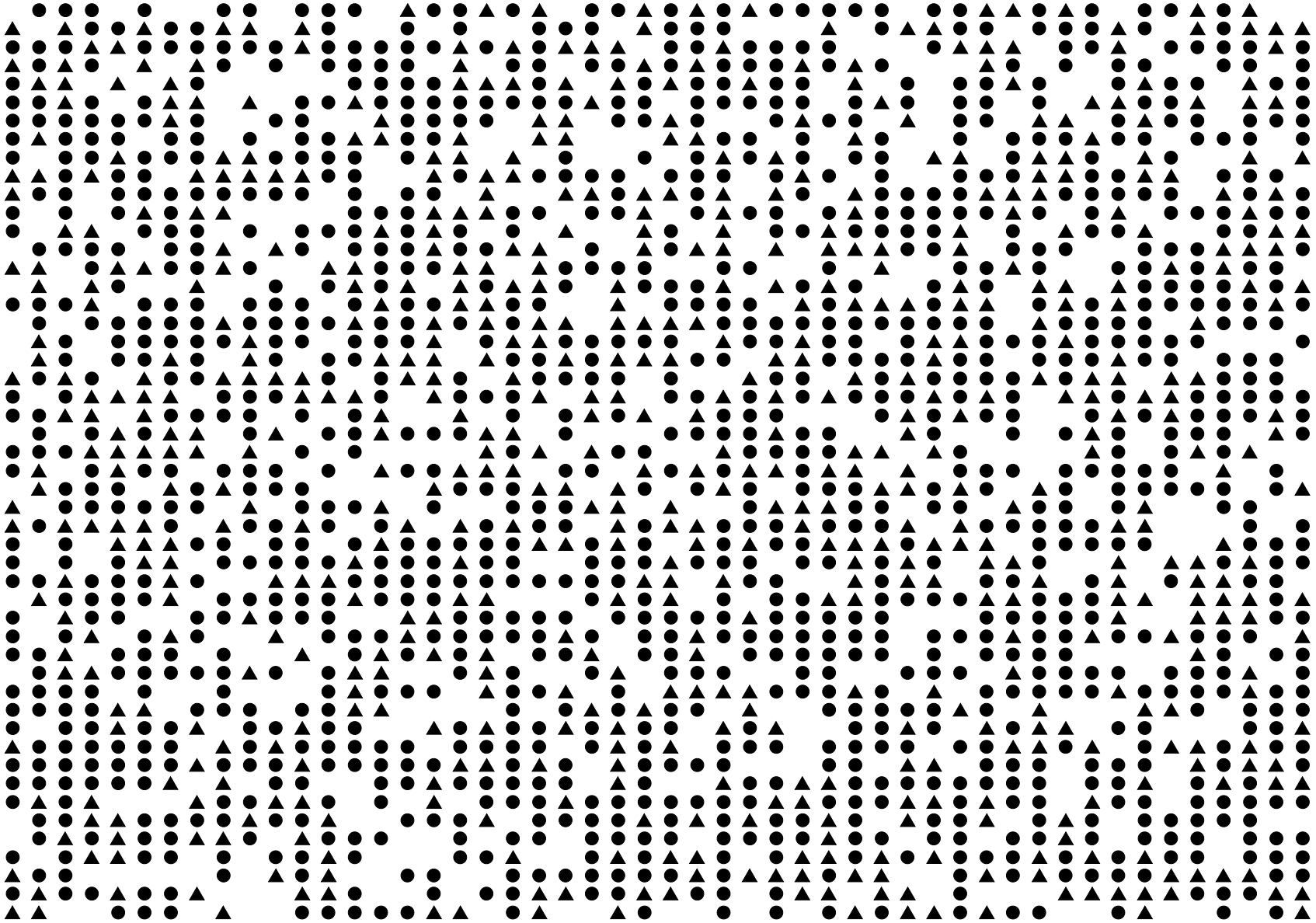
2. Use your method to estimate the number of:

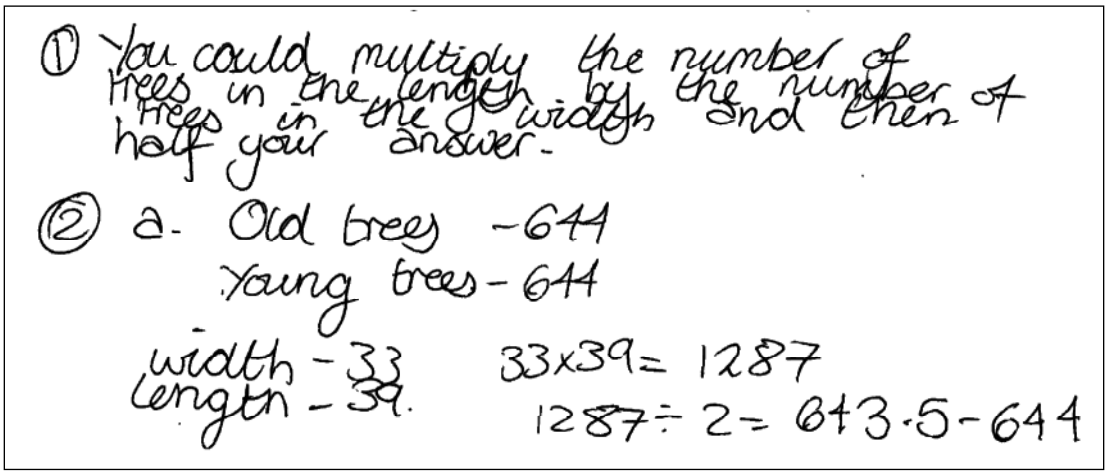
(a) Old trees.

(b) New trees.

The Tree Farm



Sample Responses to Discuss: Laura



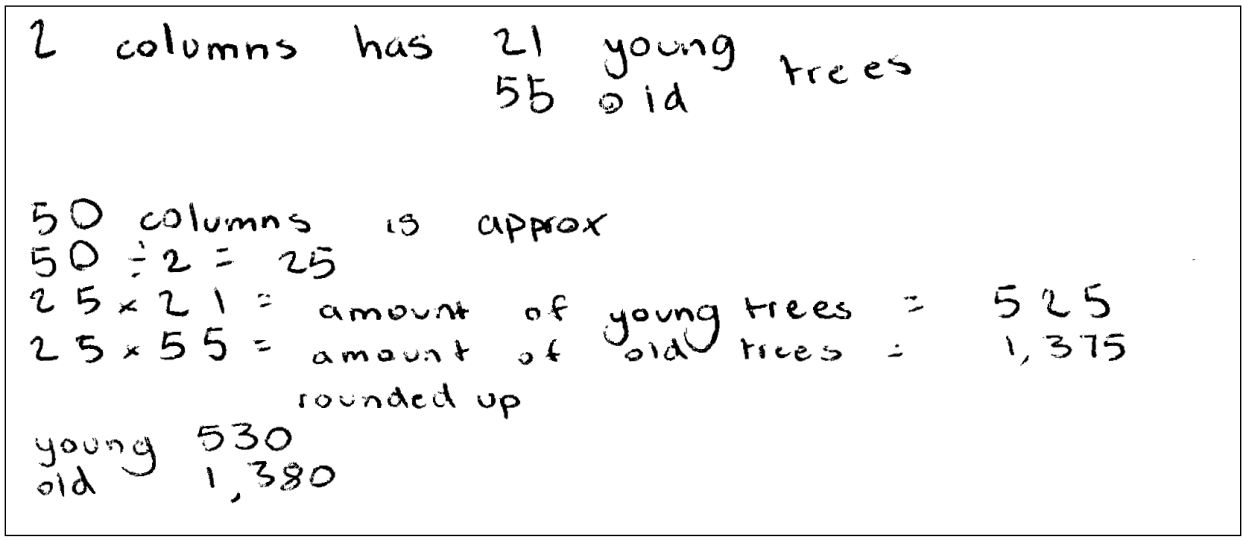
Does Laura's approach make mathematical sense? Why does she halve her answer?

What assumptions has Laura made?

In what ways could Laura's work be improved?

To help you to understand Laura's work, what question(s) could you ask her?

Sample Responses to Discuss: Wayne



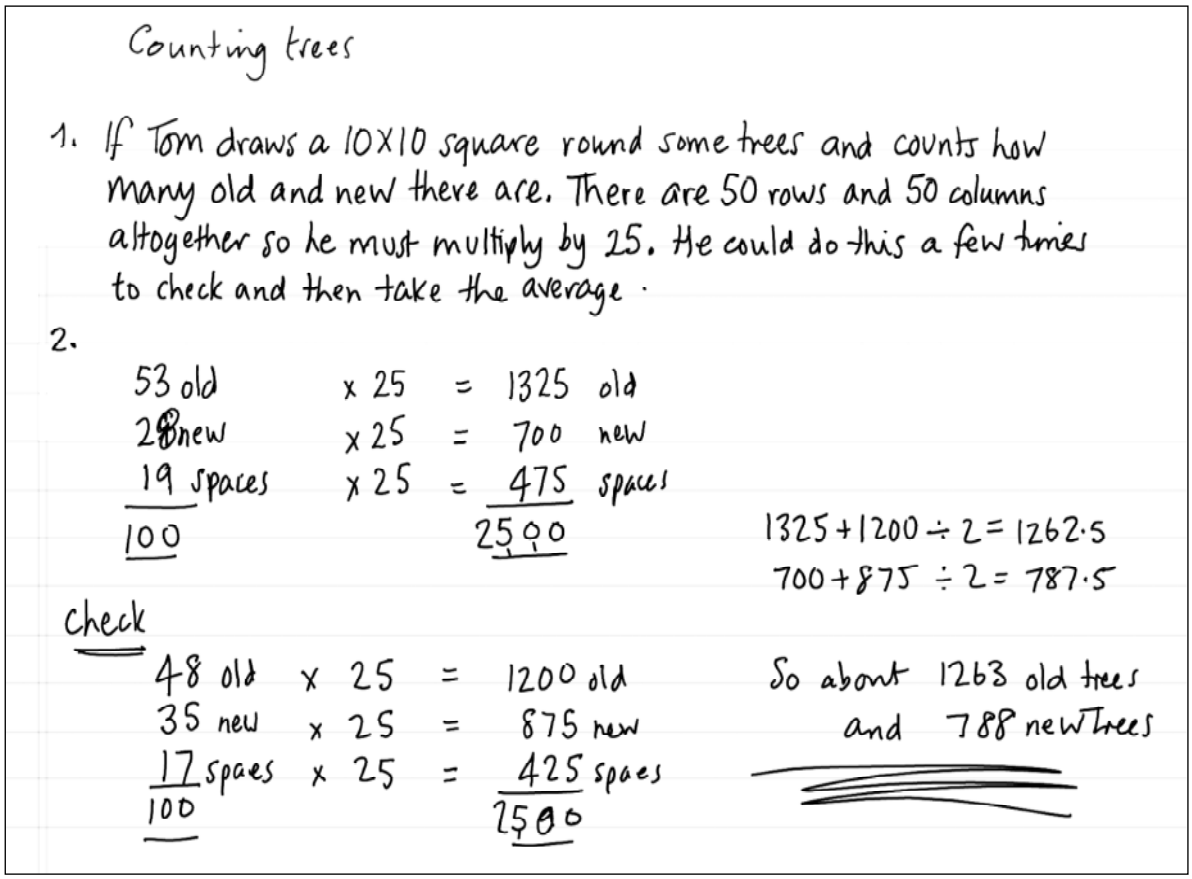
Does Wayne's approach make mathematical sense? Why has he multiplied by 25?

What assumptions has Wayne made?

In what ways could Wayne's work be improved?

To help you to understand Wayne's work, what question(s) could you ask him?

Sample Responses to Discuss: Amber



Counting trees

1. If Tom draws a 10X10 square round some trees and counts how many old and new there are. There are 50 rows and 50 columns altogether so he must multiply by 25. He could do this a few times to check and then take the average.

2.

53 old x 25 = 1325 old
28 new x 25 = 700 new
19 spaces x 25 = 475 spaces
100 2500

1325 + 1200 ÷ 2 = 1262.5
700 + 875 ÷ 2 = 787.5

Check

48 old x 25 = 1200 old
35 new x 25 = 875 new
17 spaes x 25 = 425 spaes
100 2500

So about 1263 old trees and 788 new Trees

Does Amber's approach make mathematical sense? Why has she multiplied by 25?

In what ways could Amber's work be improved?

To help you to understand Amber's work, what question(s) could you ask her?

How Did You Work?

Check the boxes and complete the sentences that apply to your work.

1. Our group work was better than my own individual work ☐

   Our joint solution was better because: .....

2. We checked our method ☐

   We checked our method by: .....

   We could check our method by: .....

3. Our method is similar to one of the sample responses ☐

   Our method is similar to: *(add name of sample response)* .....

   I prefer **our method / the sample response** method (*circle*)

   This is because .....

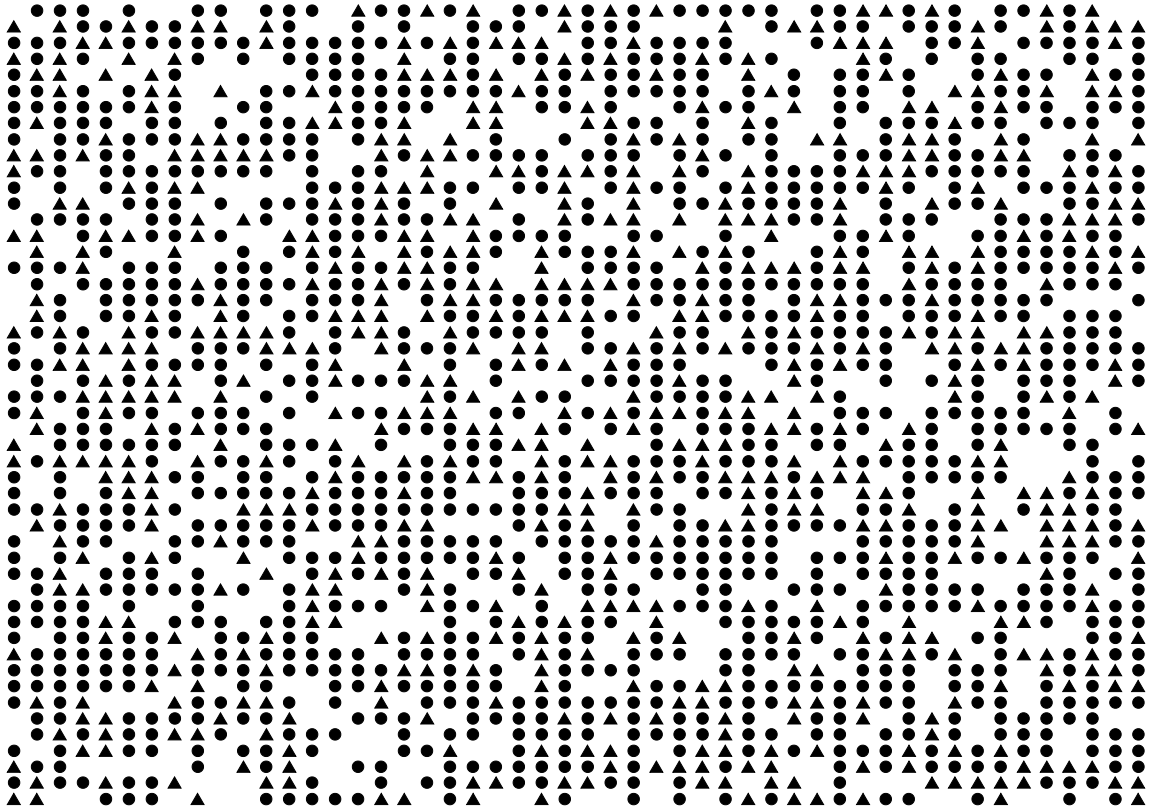
   OR Our method is different from **all** of the sample responses ☐

   Our method is different from all of the sample responses because: .....

4. In our method we assumed that: .....

Counting Trees

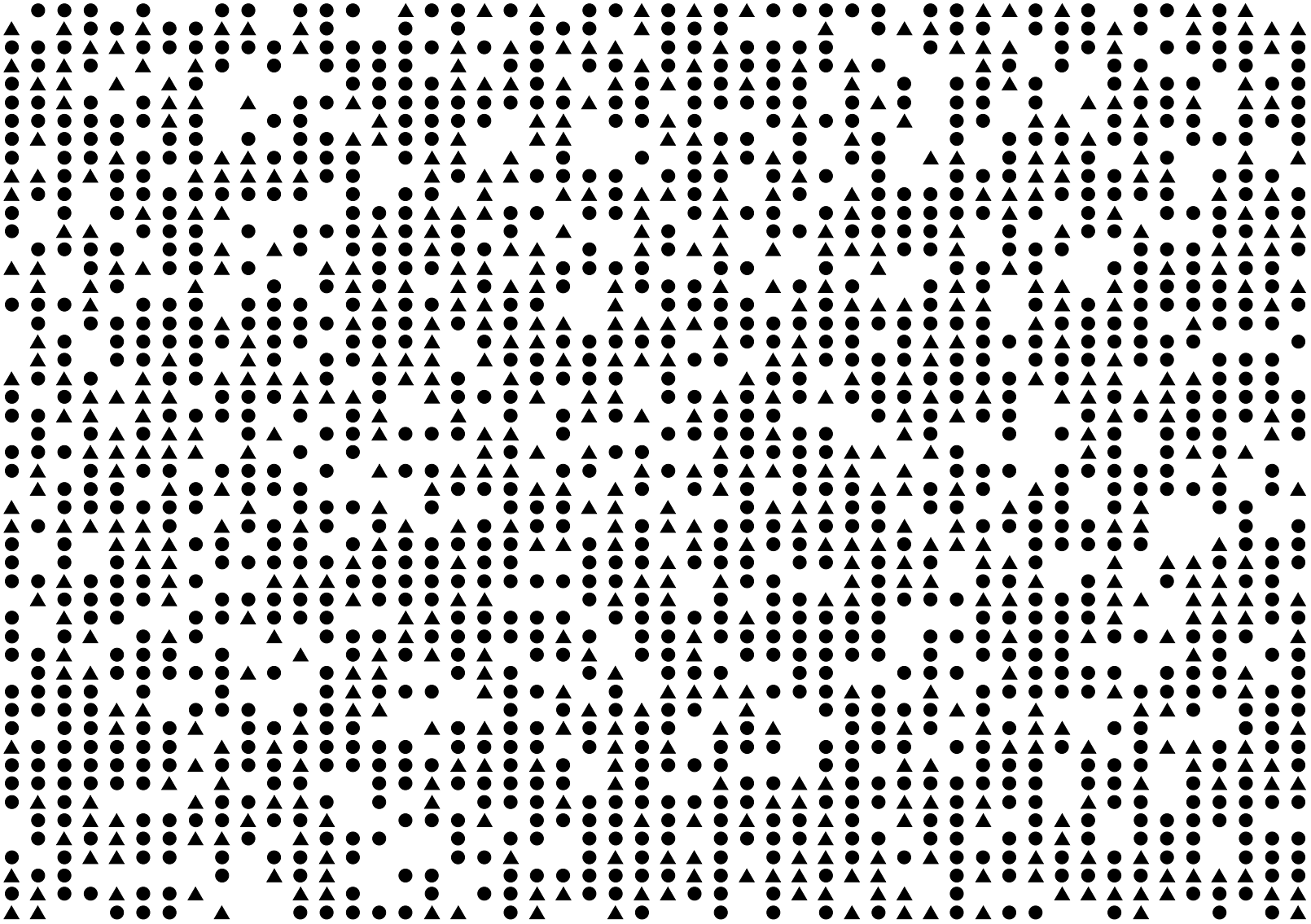
The diagram shows some trees in a tree farm.



The circles • show old trees and the triangles ▲ show young trees.

Tom wants to know how many trees there are of each type, but says it would take too long counting them all, one by one.

The Tree Farm



Collaborative Work: Joint Solution

1. Share your method with your partner(s) and your ideas for improving your individual solution.

2. Together in your group, agree on the best method for completing the problem.

3. Produce a poster, showing a joint solution to the problem.

4. Make sure that everyone in the group can explain the reasons for your chosen method, and describe any assumptions you have made.

Analyzing Sample Responses to Discuss

1. Does the approach make mathematical sense?

2. What assumptions has the student made?

3. How could the solution be improved?

4. What questions could you ask the student, to help you understand their work?

Sample Responses to Discuss: Laura

① You could multiply the number of trees in the length by the number of trees in the width and then ÷ half your answer.

② a. Old trees - 644

Young trees - 644

width - 33

length - 39

33 x 39 = 1287

1287 ÷ 2 = 643.5 - 644

Sample Responses to Discuss: Wayne

2 columns has 21 young trees
55 old

50 columns is approx
50 ÷ 2 = 25
25 × 21 = amount of young trees = 525
25 × 55 = amount of old trees = 1,375
rounded up

young 530
old 1,380

Sample Responses to Discuss: Amber

Counting trees

1. If Tom draws a 10X10 square round some trees and counts how many old and new there are. There are 50 rows and 50 columns altogether so he must multiply by 25. He could do this a few times to check and then take the average.

2.

53 old	x 25	=	1325	old
28 new	x 25	=	700	new
19 spaces	x 25	=	475	spaces
100			2500	

1325 + 1200 ÷ 2 = 1262.5

700 + 875 ÷ 2 = 787.5

Check

48 old	x 25	=	1200	old
35 new	x 25	=	875	new
17 spaces	x 25	=	425	spaces
100			2500	

So about 1263 old trees and 788 new trees

Mathematics Assessment Project

CLASSROOM CHALLENGES

This lesson was designed and developed by the
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It was refined on the basis of reports from teams of observers led by
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based on their observation of trials in US classrooms
along with comments from teachers and other users.

This project was conceived and directed for
MARS: Mathematics Assessment Resource Service
by
Alan Schoenfeld, Hugh Burkhardt, Daniel Pead, and Malcolm Swan
and based at the University of California, Berkeley

We are grateful to the many teachers, in the UK and the US, who trialed earlier versions
of these materials in their classrooms, to their students, and to
Judith Mills, Carol Hill, and Alvaro Villanueva who contributed to the design.

This development would not have been possible without the support of
Bill & Melinda Gates Foundation
We are particularly grateful to
Carina Wong, Melissa Chabran, and Jamie McKee