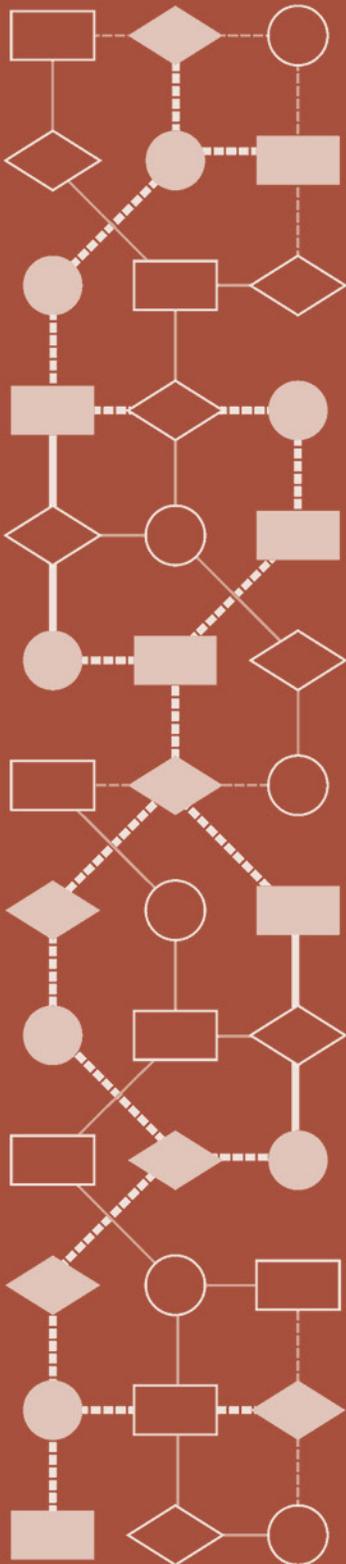


Mathematics Assessment Project  
**CLASSROOM CHALLENGES**  
A Formative Assessment Lesson

# Modeling Conditional Probabilities 1: Lucky Dip

Mathematics Assessment Resource Service  
University of Nottingham & UC Berkeley  
Beta Version

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# Modeling Conditional Probabilities 1: *Lucky Dip*

## MATHEMATICAL GOALS

This lesson unit is intended to help you assess how well students are able to:

- Understand conditional probability.
- Represent events as a subset of a sample space using tables and tree diagrams.
- Communicate their reasoning clearly.

## COMMON CORE STATE STANDARDS

This lesson relates to the following *Mathematical Practices* in the *Common Core State Standards for Mathematics*:

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.

This lesson gives students the opportunity to apply their knowledge of the following *Standards for Mathematical Content* in the CCSS:

- S-CP: Understand independence and conditional probability and use them to interpret data.
- S-MD: Calculate expected values and use them to solve problems.

## INTRODUCTION

This lesson unit is structured in the following way:

- Before the lesson, students work individually on an assessment task that is designed to reveal their current understanding and difficulties. You then review their work and create questions for students to answer in order to improve their solutions.
- At the start of the lesson, students work alone answering your questions about the same problem. Students are then grouped and engage in a collaborative discussion of the same task.
- In the same small groups, students are given sample solutions to analyze and evaluate.
- Finally, in a whole-class discussion, students explain and compare the alternative solution strategies they have seen and used.
- Finally, students review what they have learnt.

## MATERIALS REQUIRED

- Each student will need a copy of the assessment task, *Lucky Dip*, a copy of the review sheet, *How did you work?*, a mini-whiteboard, a pen, and an eraser.
- Each small group of students will need a large sheet of paper for making a poster, a felt-tipped pen, and enlarged copies of the *Student Sample Responses*.
- There are some projector resources to support whole class discussion.
- You will need a bag, and some black and white balls (or some substitute) for a class demonstration.

## TIME NEEDED

20 minutes before the lesson, a 1- hour lesson, and 10 minutes in a subsequent lesson. Exact timings will depend on the needs of the class.

## BEFORE THE LESSON

### Assessment task: *Lucky Dip* task (20 minutes)

Set this task, in class or for homework, a few days before the formative assessment lesson. This will give you an opportunity to assess the work, and to find out the kinds of difficulties students have with it. You will then be able to target your help more effectively in the follow-up lesson.

Give each student a copy of the assessment task *Lucky Dip*.

Make sure the class understands the rules of the game by demonstrating it using a bag, and some black and white balls.

*Read through the questions and try to answer them as carefully as you can.*

It is important that students, as far as possible, are allowed to answer the questions without your assistance.

Students should not worry too much if they cannot understand or do everything, because in the next lesson they will engage in a similar task, which should help them. Explain to students that by the end of the next lesson, they should expect to answer questions such as these confidently. This is their goal.

Students who sit together often produce similar answers, and then when they come to compare their work, they have little to discuss. For this reason, we suggest that when students do the task individually, you ask them to move to different seats. Then at the beginning of the formative assessment lesson, allow them to return to their usual seats. Experience has shown that this produces more profitable discussions.

### Assessing students' responses

Collect students' responses to the task. Make some notes on what their work reveals about their current levels of understanding and their different problem solving approaches.

We suggest that you do not score students' work. The research shows that this will be counterproductive, as it will encourage students to compare their scores, and will distract their attention from what they can do to improve their mathematics.

Instead, help students to make further progress by summarizing their difficulties as a series of questions. Some suggestions for these are given on the next page. These have been drawn from common difficulties observed in trials of this unit.

We recommend that you write a selection of questions on each student's work. If you do not have time, select a few questions that will be of help to the majority of students. These can be written on the board at the beginning of the lesson.

**Lucky Dip**

Dominic has devised a simple game.  
Inside a bag he places 3 black and 3 white balls. He then shakes the bag.

He asks Amy to take two balls from the bag without looking.



**Dominic**

If the two balls are the same color then you win.  
If they are different colors then I win.

OK.  
That sounds fair to me.



**Amy**



Is Amy right? Is the game fair?  
If Amy is wrong, then who is most likely to win?  
Show all your reasoning clearly.

**Common issues****Suggested questions and prompts**

<p><b>Student produces no work.</b></p>	<ul style="list-style-type: none"> <li>• Play the game twenty times, using pieces of marked paper instead of balls. Do you think the game is fair? Explain your answer.</li> </ul>
<p><b>Student assumes the game is fair.</b></p> <p>For example: The student assumes that there are only two outcomes (the balls are the same color or the balls are different colors), so the probabilities are equal.</p>	<ul style="list-style-type: none"> <li>• Suppose you labeled each ball with a different letter. What are the different combinations you can take out of the bag?</li> </ul>
<p><b>Student does not choose a suitable representation.</b></p> <p>For example: The student does not use a tree diagram, or a sample space diagram.</p>	<ul style="list-style-type: none"> <li>• Can you think of a suitable diagram that will show all the possible outcomes?</li> <li>• Can you use a sample space diagram?</li> </ul>
<p><b>Student is confused as to the nature of the “event”.</b></p> <p>For example: The student is concerned over whether it makes a difference to consider taking both balls out at once or taking them out one at a time.</p>	<ul style="list-style-type: none"> <li>• Does it make a difference whether Amy picks the balls one at a time, rather than at the same time? Explain your answer.</li> <li>• How can you show the different possible outcomes using a diagram?</li> </ul>
<p><b>Student does not recognize dependent probabilities.</b></p> <p>The student appears to assume that each ball is returned to the bag after it is selected.</p> <p>For example: The probability that the second ball is black is assumed to be independent of the choice of the first ball. So <math>P(\text{both balls black})</math> is assumed to be <math>0.5 \times 0.5</math>.</p>	<ul style="list-style-type: none"> <li>• Imagine picking a black ball from the bag. What is the probability of picking a black ball? What has changed? Now you are holding the black ball, what is the probability of picking another black ball?</li> </ul>
<p><b>Student selects the same ball twice in their table of possible outcomes.</b></p> <p>For example: The student assumes that there are <math>3 \times 3 = 9</math> ways of obtaining two black balls.</p>	<ul style="list-style-type: none"> <li>• Is it possible to select the same ball twice?</li> </ul>
<p><b>Student presents incomplete or unclear work.</b></p> <p>For example: The student does not fully label the tree diagram or the sample space diagram.</p>	<ul style="list-style-type: none"> <li>• Would someone unfamiliar with this type of work understand all your work?</li> </ul>
<p><b>Student correctly answers all the questions.</b></p> <p>Student needs an extension task.</p>	<ul style="list-style-type: none"> <li>• How many black balls and how many white balls could you put in the bag to make the game fair? Explain your answer.</li> </ul>

## SUGGESTED LESSON OUTLINE

### Individual work (10 minutes)

Return the assessment task to the students. Give each student a mini-whiteboard, a pen, and an eraser.

Begin the lesson by briefly reintroducing the problem.

If you did not add questions to individual pieces of work, write your list of questions on the board. Students are to select questions appropriate to their own work, and spend a few minutes answering them. If students struggle to identify which questions they should be considering, it may be helpful to give them a printed version of the list of questions and highlight the ones that you want them to focus on.

*Recall what we were looking at in a previous lesson. What was the task about?*

*Today we are going to work together to try to improve your initial attempts at this task.*

*I have had a look at your work, and I have some questions I would like you to think about.*

*On your own, carefully read through the questions I have written. Use the questions to help you to think about ways of improving your own work.*

*Use your mini-whiteboards to make a note of anything you think will help to improve your work.*

Slide P-1 of the projector resource outlines the rules of the game. To remind students of the rules, you could demonstrate the game using a bag and some balls.

**Lucky Dip**

Dominic has devised a simple game.

Inside a bag he places 3 black and 3 white balls. He then shakes the bag.

He asks Amy to take two balls from the bag without looking.

**Dominic**

If the two balls are the same color, then you win.

If they are different colors, then I win.

**Amy**

OK, that sounds fair to me.

Is Amy right? Is the game fair?  
If Amy is wrong, then who is most likely to win?  
Show all your reasoning clearly.

Projector Resources Modeling Conditional Probabilities 1 P-1

### Collaborative small-group work (10 minutes)

Organize the class into small groups of two or three students. Give each group a large, piece of paper, and a felt-tipped pen.

#### Deciding on a Strategy

Ask students to share their ideas about the task, and plan a joint solution.

*I want you to share your work with your group.*

*Take turns to explain how you did the task and how you now think it could be improved.*

*Listen carefully to any explanation. Ask questions if you don't understand or agree with the method. (You may want to use some of the questions I have written on the board.)*

*I want you to plan a joint approach that is better than your separate solutions.*

Once students have evaluated the relative merits of each approach ask them to write their strategy on the second side of the poster.

Slide P-2 of the projector resource, *Planning a Joint Solution*, summarizes these instructions.

**Planning a Joint Solution**

1. Take turns to explain how you did the task and how you now think it could be improved.
2. Listen carefully to explanations.
  - Ask questions if you don't understand.
  - Discuss with your partners:
    - What you like/dislike about your partners' math.
    - Any assumptions your partner has made.
    - How their work could be improved.
3. Once everyone in the group has explained their solution, plan a joint approach that is better than each of the separate solutions.
  - On the second side of your poster or paper write a couple of sentences outlining your plan.

### **Implementing the Strategy**

Students are now to write their joint solution on the front side of the poster.

While students work in small groups you have two tasks: to note different student approaches to the task, and to support student problem solving

You can then use this information to focus a whole class discussion towards the end of the lesson. For example, do students identify all the different possible events clearly? Are students using diagrams to support their answers? In particular, note any common mistakes.

Try not to make suggestions that move students towards a particular approach to the task. Instead, ask questions that help students to clarify their thinking. In particular focus on the strategies rather than the solution. Encourage students to justify their statements.

Look for any groups of students who agree amongst themselves on an incorrect answer or justification. You could ask these students to work with another group, to compare solutions and prompt revision.

You may want to use the questions in the *Common issues* table to support your own questioning. If the whole class is struggling on the same issue, you could write one or two relevant questions on the board and hold a brief whole-class discussion. You could also give any struggling students one of the sample responses.

### **Whole-class discussion (10 minutes)**

You may want to hold a brief whole-class discussion. Have students solved the problem using a variety of methods? Or have you noticed some interesting ways of working or some incorrect methods, if so, you may want to focus the discussion on these. Equally, if you have noticed different groups using similar strategies but making different assumptions you may want to compare solutions.

### **Collaborative analysis of Sample Student Responses (15 minutes)**

After students have had enough time to attempt the problem, give each group copies of the three *Sample student responses*, and ask for written comments. This task gives students the opportunity to evaluate a variety of approaches to the task.

*You are now going to look at three solutions to the task.*

*Imagine you are the teacher. Write down your comments on each piece of work.*

*Try to explain what the student has done.*

*What mistakes have been made?*

*What isn't clear about the work?*

Slide P-3 of the projector resource, *Evaluating Sample Student Work*, describes how students should work together:

**Evaluating Student Sample Responses**

1. Imagine you are the teacher and have to assess the student work.
2. Take turns to work through a students' solution.
  - Write your answers on your mini-whiteboards.
3. Explain your answer to the rest of the group.
4. Listen carefully to explanations.
  - Ask questions if you don't understand.
5. Once everyone is satisfied with the explanations, write the answers below the students' solution.
  - Make sure the student who writes the answers is not the student who explained them.

During the paired work, support the students as in the first collaborative activity.

Note similarities and differences between the approaches seen in the *Sample responses* and those students took in the small-group work. Also, check to see which methods students have difficulties in understanding. This information can help you focus the next activity, a whole-class discussion.

**Whole-class discussion: comparing different approaches (15 minutes)**

Hold a whole-class discussion to consider the different approaches used in the sample work. Focus the discussion on parts of the task students found difficult. Ask the students to compare the different solution methods.

*Which approach did you like best? Why?*

*Which approach did you find most difficult to understand? Why?*

To support the discussion, you may want to use the projector resource to display the sample work.

**Anna's** work appears intuitively correct.

She assumes that there are only two outcomes (that the two balls are the same color or that they are different colors), so that the probabilities are equal.

Anna does not take into account the changes in probabilities once a ball is removed from the bag and not replaced.

Amy could select  
Black + black  
Black + white  
White + black  
White + white

There are 2 when the balls  
are the same color + 2  
when the balls are different

THE GAME IS FAIR

**Ella** draws a sample space in the form of an organized table.

Ella clearly presents her work, however she makes the mistake of including the diagonals. This means the same ball is selected twice. This is not possible, as the balls are not replaced.

		2nd ball						
		B1	B2	B3	W1	W2	W3	
1st ball	B1	Amy	A	A	D	D	D	
	B2	A	A	A	D	D	D	
	B3	A	A	A	D	D	D	
		W1	D	D	D	A	A	A
		W2	D	D	D	A	A	A
		W3	D	D	D	A	A	A

There are 36 equally likely outcomes.  
 Amy wins 18 times  
 Dominic wins 18 times  
 So the game is fair.

**Jordan** uses a tree diagram to show the possible outcomes when taking two balls from the bag.

Jordan's work is difficult to follow. He does not label the branches of the tree.

Jordan does not take into account that the first ball is not replaced. When selecting the second ball there are only 5 balls in the bag, so these probability fractions should all have a denominator of 5.

2 same color =  $\frac{6}{36} + \frac{6}{36} = \frac{12}{36} = \frac{1}{3}$   
 2 different color =  $\frac{9}{36} + \frac{9}{36} = \frac{18}{36} = \frac{1}{2}$

- Where does the denominator of 6 come from?*
- What is the sum of all final probabilities?*
- What does this tell you about Jordan's work?*  
*[He has made a mistake.]*

**Review solutions to Lucky Dip (10 minutes)**

Give out the sheet *How Did You Work?* and ask students to complete this questionnaire.

Some teachers set this task as homework.

The questionnaire should help students review their progress.

If you have time you may also want to ask your students to read through their original solution and using what they have learned, attempt the task again. In this case, give each student a fresh blank copy of the assessment *Lucky Dip*.

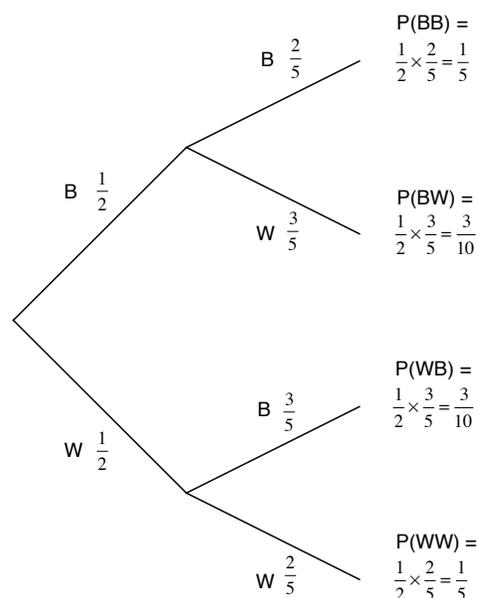
## SOLUTIONS

### Lucky Dip

Amy is wrong: the game is not fair. In the sample space diagram below, the black balls are labeled  $B_1, B_2, B_3$  and the white balls are labeled  $W_1, W_2, W_3$ . Each cell shows one possible, equally likely outcome. The diagonal doesn't show possible outcomes because the same ball cannot be taken out twice.

Amy wins wherever there is a ✓. Dominic wins wherever there is a x. This shows that the probability of Amy winning is  $\frac{12}{30} = \frac{2}{5}$  and the probability of Dominic winning is  $\frac{18}{30} = \frac{3}{5}$ . An alternative representation is the tree diagram.

		First selection					
		$B_1$	$B_2$	$B_3$	$W_1$	$W_2$	$W_3$
Second selection	$B_1$	-	✓	✓	x	x	x
	$B_2$	✓	-	✓	x	x	x
	$B_3$	✓	✓	-	x	x	x
	$W_1$	x	x	x	-	✓	✓
	$W_2$	x	x	x	✓	-	✓
	$W_3$	x	x	x	✓	✓	-



Some students think of the event being modeled as **picking two balls simultaneously**. In that case, the sample space diagram (with labels first selection, second selection) and the probability tree (which again shows a sequence of events) may seem less appropriate. The student may therefore decide to not distinguish between  $B_1B_2$  and  $B_2B_1$ . The resulting sample space diagram will be just the upper (or lower) half of the sample space diagram shown above. The resulting probabilities however will remain unaffected.



## Student Sample Responses: Anna

Amy could select  
Black + black  
Black + white  
White + black  
White + white

There are 2 when the balls  
are the same color + 2  
when the balls are different

THE GAME IS FAIR

Explain what the student has done. ....

.....

What isn't clear about her work? .....

.....

What mistakes has she made? .....

.....

## Student Sample Responses: Ella

		2nd ball					
		B1	B2	B3	W1	W2	W3
1st ball	B1	Amy	A	A	D	D	D
	B2	A	A	A	D	D	D
	B3	A	A	A	D	D	D
	W1	D	D	D	A	A	A
	W2	D	D	D	A	A	A
	W3	D	D	D	A	A	A

There are 36 equally likely outcomes.

Amy wins 18 times

Dominic wins 18 times

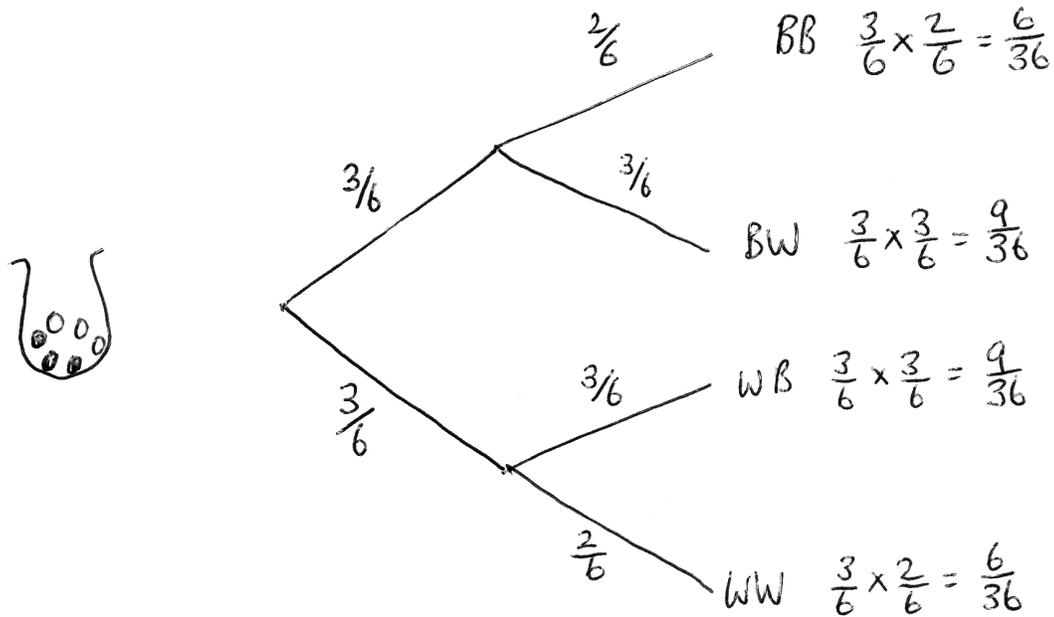
So the game is fair.

Explain what the student has done.

What isn't clear about her work?

What mistakes has she made?

## Student Sample Responses: Jordan



$$2 \text{ same color} = \frac{6}{36} + \frac{6}{36} = \frac{12}{36} = \frac{1}{3}$$

$$2 \text{ different color} = \frac{9}{36} + \frac{9}{36} = \frac{18}{36} = \frac{1}{2}$$

Explain what the student has done. ....

.....

.....

What isn't clear about his work? .....

.....

.....

What mistakes has he made? .....

.....

.....

# How Did You Work?

Mark the boxes and complete the sentences that apply to your work.

<p>1. Our group work was better than my own work</p> <p>Our group solution was better because</p> <p>.....</p> <p>.....</p> <p>.....</p> <p>.....</p>	<input type="checkbox"/>
<p>2. Our solution is similar to one of the sample responses</p> <p>Our solution is similar to <i>(add name of the student)</i></p> <p>I prefer <b>our solution</b> / <b>the student's solution</b> <i>(circle)</i></p> <p>because .....</p> <p>.....</p> <p>.....</p>	<input type="checkbox"/>
<p><b>OR</b></p>	
<p>Our solution is different from <b>all</b> of the sample responses</p> <p>Our solution is different from all of the sample responses because</p> <p>.....</p> <p>.....</p> <p>.....</p> <p>.....</p>	<input type="checkbox"/>
<p>3. What advice would you give a student new to this task about potential pitfalls?</p> <p>.....</p> <p>.....</p> <p>.....</p> <p>.....</p>	

# Lucky Dip

Dominic has devised a simple game.

Inside a bag he places 3 black and 3 white balls.  
He then shakes the bag.



He asks Amy to take two balls from the bag without looking.

**Dominic**



If the two balls are the same color, then you win.

If they are different colors, then I win.

**Amy**



OK, that sounds fair to me.

Is Amy right? Is the game fair?  
If Amy is wrong, then who is most likely to win?  
Show all your reasoning clearly.

# Planning a Joint Solution

1. Take turns to explain how you did the task and how you now think it could be improved.
2. Listen carefully to explanations.
  - Ask questions if you don't understand.
  - Discuss with your partners:
    - What you like/dislike about your partners' math.
    - Any assumptions your partner has made.
    - How their work could be improved.
3. Once everyone in the group has explained their solution, plan a joint approach that is better than each of the separate solutions.
  - On the second side of your poster or paper write a couple of sentences outlining your plan.

# Evaluating Student Sample Responses

1. Imagine you are the teacher and have to assess the student work.
2. Take turns to work through a students' solution.
  - Write your answers on your mini-whiteboards.
3. Explain your answer to the rest of the group.
4. Listen carefully to explanations.
  - Ask questions if you don't understand.
5. Once everyone is satisfied with the explanations, write the answers below the students' solution.
  - Make sure the student who writes the answers is not the student who explained them.

# Anna's Response

Amy could select  
Black + black  
Black + white  
White + black  
White + white

There are 2 when the balls  
are the same color + 2  
when the balls are different

THE GAME IS FAIR

# Ella's Response

		2nd ball						
		B1	B2	B3	W1	W2	W3	
1st ball	B1	Amy	A	A	D	D	D	
	B2	A	A	A	D	D	D	
	B3	A	A	A	D	D	D	
		W1	D	D	D	A	A	A
		W2	D	D	D	A	A	A
		W3	D	D	D	A	A	A

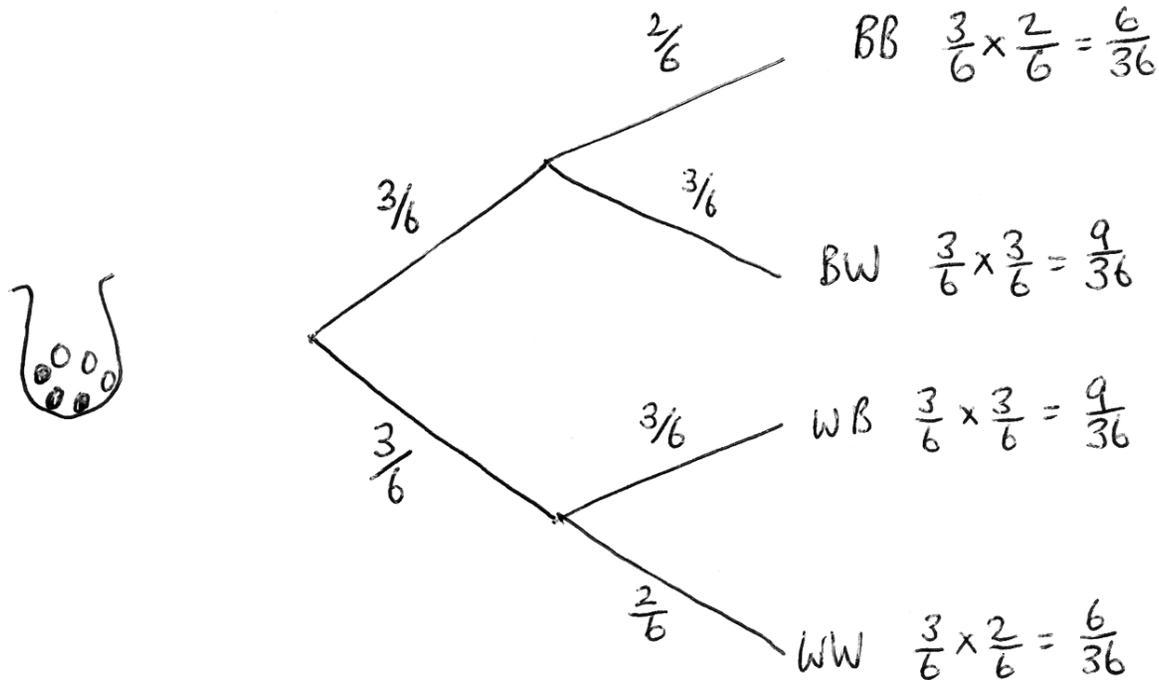
There are 36 equally likely outcomes.

Amy wins 18 times

Dominic wins 18 times

So the game is fair.

# Jordan's Response



$$2 \text{ same color} = \frac{6}{36} + \frac{6}{36} = \frac{12}{36} = \frac{1}{3}$$

$$2 \text{ different color} = \frac{9}{36} + \frac{9}{36} = \frac{18}{36} = \frac{1}{2}$$

Mathematics Assessment Project  
**CLASSROOM CHALLENGES**

This lesson was designed and developed by the  
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It was refined on the basis of reports from teams of observers led by  
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based on their observation of trials in US classrooms  
along with comments from teachers and other users.

This project was conceived and directed for  
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and based at the University of California, Berkeley

We are grateful to the many teachers, in the UK and the US, who trialed earlier versions  
of these materials in their classrooms, to their students, and to  
Judith Mills, Carol Hill, and Alvaro Villanueva who contributed to the design.

This development would not have been possible without the support of  
**Bill & Melinda Gates Foundation**  
We are particularly grateful to  
Carina Wong, Melissa Chabran, and Jamie McKee

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