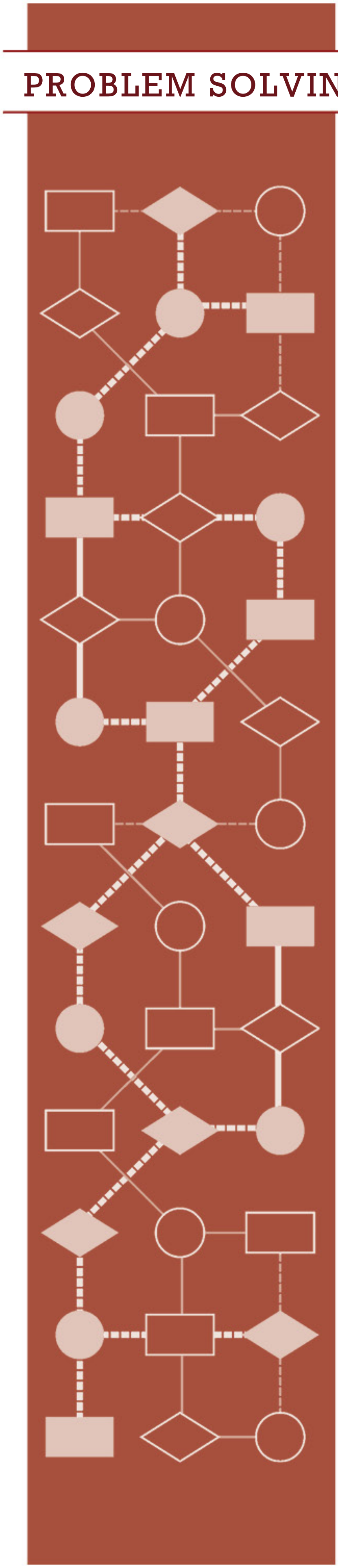
PROBLEM SOLVING



Mathematics Assessment Project
CLASSROOM CHALLENGES
A Formative Assessment Lesson

Modeling: Having Kittens

Mathematics Assessment Resource Service
University of Nottingham & UC Berkeley
Beta Version

Modeling: Having Kittens

MATHEMATICAL GOALS

This lesson unit is intended to help you assess how well students are able to:

- Interpret a situation and represent the constraints and variables mathematically.
- Select appropriate mathematical methods to use.
- Make sensible estimates and assumptions.
- Investigate an exponentially increasing sequence.
- Communicate their reasoning clearly.

COMMON CORE STATE STANDARDS

This lesson relates to the following *Mathematical Practices* in the *Common Core State Standards for Mathematics*:

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.

This lesson gives students the opportunity to apply their knowledge of the following *Standards for Mathematical Content* in the *Common Core State Standards for Mathematics*:

N-Q: Reason quantitatively and use units to solve problems.
F-LE: Construct and compare linear, quadratic, and exponential models and solve problems.

INTRODUCTION

This lesson is designed to help students develop strategies for modeling. Note that a video of this lesson is available in the professional development materials.

- Before the lesson, students attempt the problem individually. You then review their work and write questions to help students improve their solutions.
- At the start of the lesson, students work individually answering questions about the same problem. In small groups, students then work collaboratively on the task. In the same small groups, students evaluate sample solutions. In a whole-class discussion, students explain and compare the alternative solution strategies they have seen and used.
- In a follow-up lesson, students review what they have learned.

MATERIALS REQUIRED

- Each individual student will need a calculator, a copy of the assessment task *Having Kittens*, the questionnaire *How Did You Work,* a mini-whiteboard, an eraser, and a pen.
- Each small group will need a copy of all the *Sample Responses to Discuss,* a large sheet of paper for making a poster, and felt-tipped pens. Graph paper should be kept in reserve and used only when requested.
- There are some projector resources to help you with whole-class discussions. Spreadsheet software might also be helpful if available.

TIME NEEDED

20 minutes before the lesson, a 90-minute lesson, and 10 minutes in a follow-up lesson (or for homework). Timings given are only approximate. Exact timings will depend on the needs of your class.

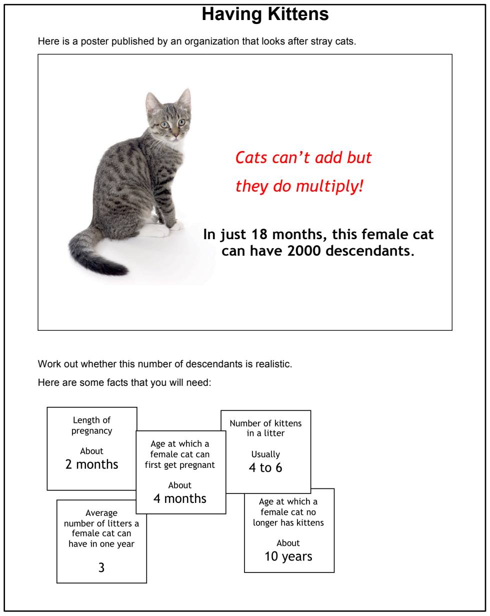
BEFORE THE LESSON

Assessment task: *Having Kittens* (20 minutes)

Ask students to do this task in class or for homework a day or more before the formative assessment lesson. This will give you an opportunity to assess their work, and to find out the kinds of difficulties students have with it. You should then be able to target your help more effectively in the follow-up lesson.

Give out the task *Having Kittens*.

> *This is a poster made by a cats' charity, encouraging people to have their cats spayed so they can't have kittens.*
>
> *The activity is about what happens if you don't have your cat spayed. Your task is to decide whether the statement on the poster is correct.*
>
> *Is it realistic that one female cat would produce 2000 descendants in 18 months?*
>
> *You are given some facts about cats and kittens that will help you decide.*



Having Kittens

Here is a poster published by an organization that looks after stray cats.

Cats can't add but they do multiply!

In just 18 months, this female cat can have 2000 descendants.

Work out whether this number of descendants is realistic.

Here are some facts that you will need:

Length of pregnancy — About 2 months

Age at which a female cat can first get pregnant — About 4 months

Number of kittens in a litter — Usually 4 to 6

Average number of litters a female cat can have in one year — 3

Age at which a female cat no longer has kittens — About 10 years

It is important that, as far as possible, students are allowed to answer the questions without assistance.

Students who sit together often produce similar answers, and then when they come to compare their work, they have little to discuss. For this reason, we suggest that when students do the task individually, you ask them to move to different seats. Then at the beginning of the formative assessment lesson, allow them to return to their usual seats. Experience has shown that this produces more profitable discussions.

Assessing students' responses

Collect students' responses to the task. Make some notes on what their work reveals about their current levels of understanding, and their different problem solving approaches. The purpose of doing this is to forewarn you of issues that will arise during the lesson itself, so that you may prepare carefully.

We suggest that you do not score students' work. The research shows that this will be counterproductive, as it will encourage students to compare their scores and will distract their attention from what they can do to improve their mathematics.

Instead, help students to make further progress by summarizing their difficulties as a series of questions. Some suggestions for these are given on the next page. These have been drawn from common difficulties observed in trials of this lesson unit.

We suggest that you write a list of your own questions, based on your own students' work, using the ideas below. You may choose to write questions on each student's work. If you do not have time to do this, just select a few questions that will be of help to the majority of students. These can be written on the board at the beginning of the lesson. You may also want to note students with a particular difficulty, so that you can ask them about that issue in the formative lesson.

Common issues:	**Suggested questions and prompts:**
Student has difficulty getting started	• What do you know? • What do you need to find out? • Can you describe in words what happens during the first five months?
Student does not develop a suitable representation for the problem	• Can you make a diagram or table to show what is happening? • Can you show time elapsing on your diagram? • How can you show which kittens are descended from which? • How can you show the numbers of kittens at each month?
Student's work is unsystematic	• Can you think of a way of breaking the task into manageable chunks? • Could you start by just looking at the litters from the first cat? Then what would you do after that? • Can you now look systematically at what happens to her kittens? And their kittens?
Student develops a partial model For example: The student only considers litters from the original cat. Or: The student only considers the first litter at each generation.	• Do you think the first litter of kittens will have time to grow and have litters of their own? Then what about their kittens? • Do you think that the first cat will have time to have more than litter? What about her kittens?
Student does not make assumptions explicit For example: The student does not write that she has decided that all the litters occur in the first six months of the year. Or: The student does not state that she is assuming all the kittens are female.	• What have you assumed here?
Student makes unreasonable assumptions For example: The student decides that all the kittens born in a year are born at the **beginning** of that year.	• Is your assumption that [all these kittens are born at the beginning of the year] reasonable?
Student has made a successful attempt	• Do you think that your work gives the maximum possible number of kittens? How can you be sure? • Can you find a different way of presenting your analysis? Which way is clearest? • Do you think your answer is an overestimate or underestimate? Why? Can you suggest reasonable bounds for your estimate?

SUGGESTED LESSON OUTLINE

Introduction: *Cats and Kittens* (10 minutes)

Begin the lesson by briefly reintroducing the problem:

> *Do you remember the problem I asked you to do last time? I have had a look at your work and I have some questions on it.*
>
> *Today you are going to work together to improve your initial attempts.*
>
> *First, though, I would like you to work individually again. Read through the questions I have written about your work.*
>
> *Use your mini-whiteboard to note answers to these questions.*

It is helpful to ask pupils to write their ideas a mini-whiteboard using a felt-tipped pen, as you will be able to monitor their work more easily. This will also help pupils to share their ideas easily later in the lesson.

Collaborative activity: Producing a joint solution (20 minutes)

The time given for this activity is an approximate guide. You may find your class would benefit from spending more (or less) time on this activity.

Organize the class into small groups of two or three students and give each group a large sheet of paper and a felt-tipped pen. Ask students to try the task again, this time combining their ideas.

> *I want you to work in groups now.*
>
> *Your task is to produce a solution that is better than your individual solutions.*
>
> *Take turns to explain how you did the task and how you now think it could be improved, then put your individual work aside. Try to produce a joint solution to the problem.*

While students work in small groups you have two tasks: to note different student approaches to the task, and to support student problem solving. You can then use this information to focus a whole-class discussion towards the end of the lesson.

Note different student approaches to the task

In particular, notice any difficulties that students encounter with what they are doing, and the ways they justify and explain to each other. Some students may take a diagrammatic or graphical approach, while others may take a more numerical one. A few students may develop an algebraic approach. Notice also if students monitor their own progress. If there is lack of progress are they prepared to improve or change strategy? What assumptions do students make? Are they aware of these assumptions and the effect they have on their solution? Support student problem solving

Support student problem solving

If a student struggles to get started, encourage them to ask a specific question about the task. Articulating the problem in this way can sometimes offer a direction to pursue that was previously overlooked. However, if the student needs their question answered, ask another member of the group for a response.

> *What is known and what is unknown?*
>
> *What are you asked to find out?*
>
> *What kind of representation will help you tackle this problem?*

Try not to make suggestions that move students towards a particular approach to this task. Instead, ask questions that help students to clarify their thinking and encourage checking:

Can you set out your work using a table or diagram? What would be a good way?

How many cats/kittens will there be after 6 months? 12 months? ….

What assumptions have you made?

How can you check your solution?

Do you think there is just one solution?

You may also want to use the questions in the *Common issues* table.

If the whole class is struggling on the same issue, write relevant questions on the board. You could also ask students who performed well on the assessment to help struggling students. If, after several minutes, students are having difficulty making any progress at all you could hand out one of the *Sample Responses to Discuss* to get them started.

Ask each group of students you visit to review their progress.

What is your strategy for solving this problem?

What do you know now that you did not before?

Are you going to continue with this strategy?

Are there any other approaches you could try?

The purpose of these questions is to help students learn to track and review their problem solving strategies. It is important you ask these review questions of students who are and are not following what you know to be productive approaches, whether or not they are stuck. Otherwise, students will learn that your questions are really a prompt to switch strategy!

Checking posters (10 minutes)

After students have had sufficient time to attempt the problem, ask one student from each group to visit another group's poster.

If you are visiting another group, read through their work. If their work makes sense, explain it in your own words. If the work does not make sense to you, ask for clarification.

If you are staying at your desk, either carefully listen to the explanation and check it matches your own thinking or answer the visiting students' questions.

You may then want to consider improving your poster.

Sharing different approaches (20 minutes)

If you think students have produced a variety of representations of the task and/or made a range of assumptions you may want to now hold a whole class discussion. If you have noticed some interesting ways of working or some incorrect solutions you may want to focus the discussion on these. Equally, if you have noticed different groups using similar strategies but making different assumptions you may want to compare answers.

Ask other pupils to comment on:

- Did they choose a good method for representing the situation?
- Did they make sensible assumptions?
- Is the reasoning correct? Are the calculations accurate?

- Are the conclusions sensible?
- Was the reasoning easy to understand and follow?

Then give each group copies of the *Sample Responses to Discuss,* and ask for written comments. This gives students the opportunity to see further representations and discuss the assumptions made in each case.

Imagine you are the teacher and have to assess this work.

What has each student done correctly?

What assumptions have they made?

How can their work be improved?

You may decide there is not enough time for each group to work through all three pieces of work. In that case, be selective about which you hand out. For example, groups that have forgotten that a cat may have more than one litter could be given Alice's work, while groups that have forgotten that the kittens from each litter can go on to have their own litters could be given Wayne's work.

Whole-class discussion: Comparing different approaches (20 minutes)

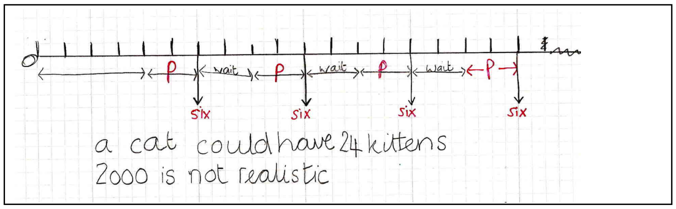
Organize a discussion about what has been learned. Discuss some of the different approaches used in the sample work and ask students to comment on their strengths and weaknesses. You may also want to compare students' own work with the sample student work.

Did any group use a similar method to Alice, Wayne or Ben?

What was same about the work? What was different about the work?

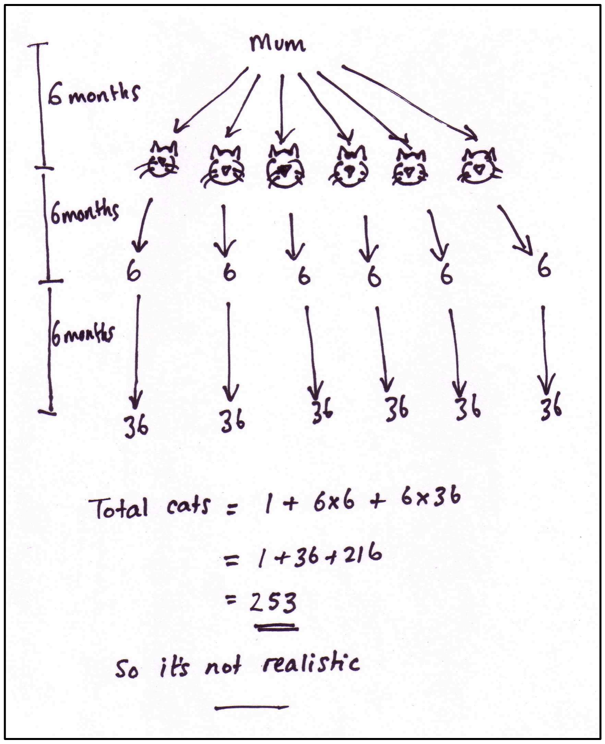
You may want to use Slides P-3, 4 and 5 of the projector resource and the questions in the *Common issues* table to support the discussion.

Alice chose to represent the task using a timeline. She has only considered the number of kittens from the original cat. She has used some of the given information correctly, and has assumed that 6 cats are born at regular intervals. She has forgotten that these kittens can also have litters of their own. She has not described her reasoning and assumptions.



a cat could have 24 kittens
2000 is not realistic

Wayne has assumed that the mother has six kittens after 6 months, and has considered succeeding generations. He has, however, forgotten that each cat may have more than one litter. He has shown the timeline clearly. Wayne doesn't explain where the 6-month gaps have come from.



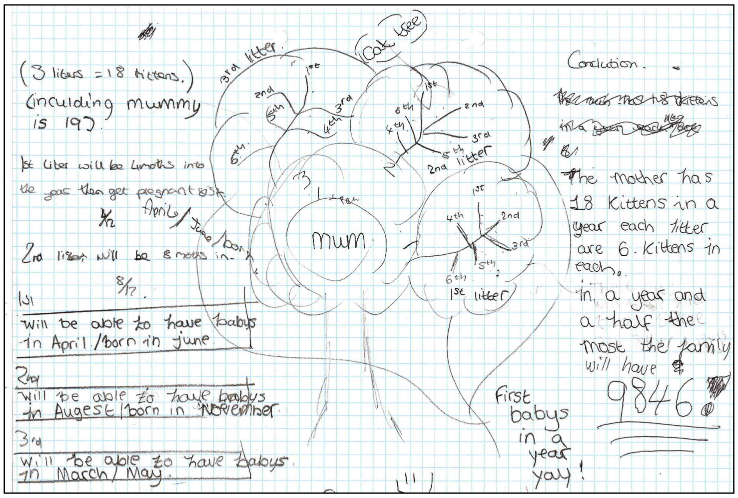
Total cats = 1 + 6x6 + 6x36

= 1+36+216

= 253

So it's not realistic

Ben has decided to draw a 'cat tree', and has tried to control for time (with some errors). The value 9846 is not explained and does not follow from the reasoning since, again, only the kittens from the original cat are considered. The number of kittens per litter is made explicit. Ben has included more explanation than Alice and Wayne.



(3 liters = 18 kittens.) (including mummy is 19)

1st liter will be 4moths into the year then get pregnant 8st April/June/born

4/12

2nd liter will be 8 moths in.

8/12.

1st

Will be able to have babys in April/born in June

2nd

Will be able to have babys in Augest/born in November

3rd

Will be able to have babys in March/May.

Conclution.

~~The mother has 18 kittens in a year each litter~~

The mother has 18 Kittens in a year each litter are 6. Kittens in each.

in a year and a half the most the family will have 9846

Review solutions to the assessment task (10 minutes)

Ask students to complete the review questionnaire *How Did You Work*.

The questionnaire may help student monitor and review their progress during and at the end of an activity.

Some teachers set this task as homework.

SOLUTIONS

There are many possible solutions to this problem, depending on the assumptions made.

In a good solution, the student will make assumptions explicit, and the chosen method of representation will involve a timeline and record the original cat and all her descendants. The exponential growth will be evident, with both the successive litters from each cat and successive generations considered.

The diagrams below are similar attempts to represent the number of kittens born in each month over an 18-month period.

These are the assumptions made for the first diagram:

- The first cat is already adult and gives birth in month 0.
- Cats become pregnant as soon as possible (at the age of 6 months).
- Litters are spread as evenly as possible across the year (so there is a litter every 4 months).
- Each litter contains six kittens.
- All of these kittens are female and have offspring of their own.
- None of the offspring die.

For the second diagram, it is still assumed that:

- Cats become pregnant as soon as possible (at the age of 6 months).
- Litters are spread as evenly as possible across the year (so there is a litter every 4 months).
- All of the kittens are female and have offspring of their own.
- None of the offspring die.

However, it is now assumed that:

- The first cat gets pregnant in month 0, and so gives birth in month 2.
- Each litter contains just three (female) kittens.

There are thus two major differences in the assumptions made:

- In the first diagram, the cat is assumed to give birth in month 0 (the first pregnancy is not counted as part of the 18-month period). In the second diagram, the cat is assumed to get pregnant in month 0, so she gives birth in month 2.
- In the first diagram, the cat is assumed to have six kittens in a litter. In the second, she has only three kittens per litter.

Diagram 1

Generation of kittens

Time in months

	0	1	2	2	2	2	3	3	3	4	Cum.total
0	1	6									7
1											7
2											7
3											7
4		6									13
5											13
6			36								49
7											49
8		6									55
9											55
10			36	36							127
11											127
12		6					216				349
13											349
14			36	36	36						457
15											457
16		6					216	216	216		1111
17											1111
18			36	36	36	36				1296	2551

Diagram 2

Generation of kittens

Time in months

	0	1	2	2	2	3	3	Cum. total
0	1							1
1								1
2		3						4
3								4
4								4
5								4
6		3						7
7								7
8			9					16
9								16
10		3						19
11								19
12			9	9				37
13								37
14		3				27		67
15								67
16			9	9	9			94
17								94
18		3				27	27	151

The number of descendants of the cat is much greater in the first diagram than in the second. Changing the set of assumptions has a significant effect on the outcome.

In both of the diagrams all the kittens are assumed to be female. It might be more reasonable to assume that about one-half of each litter is male, but counting of descendants then becomes problematic. How would we count the descendants of male cats? Each male kitten might perhaps make many females pregnant in a short time, so that many more descendants of the first cat could result.

The statement made by the vet appears to be reasonable, and might even be a conservative estimate, given the problem of descendants from male kittens.

Having Kittens

Here is a poster published by an organization that looks after stray cats.

Cats can't add but

they do multiply!

In just 18 months, this female cat can have 2000 descendants.

Figure out whether this number of descendants is realistic.

Here are some facts that you will need:

Length of pregnancy

About
2 months

Age at which a female cat can first get pregnant

About
4 months

Number of kittens in a litter

Usually
4 to 6

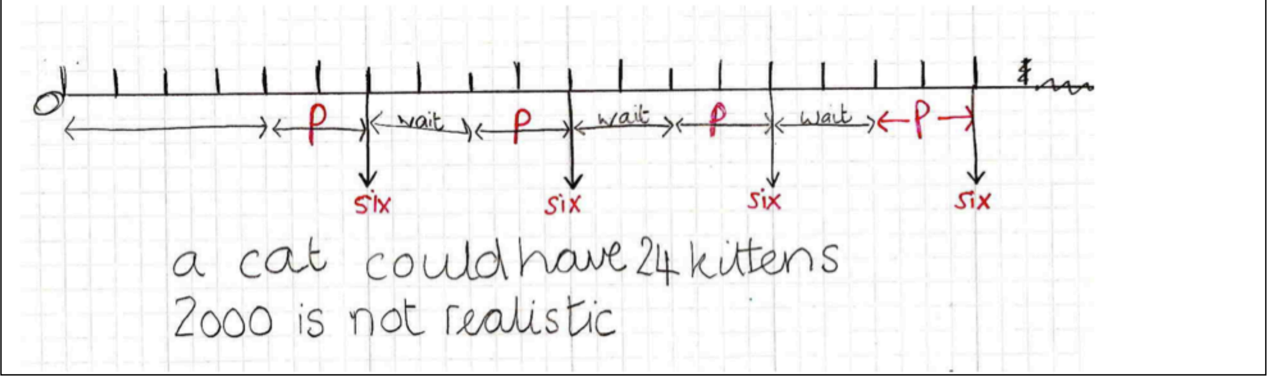
Average number of litters a female cat can have in one year

3

Age at which a female cat no longer has kittens

About
10 years

Sample Responses to Discuss: Alice



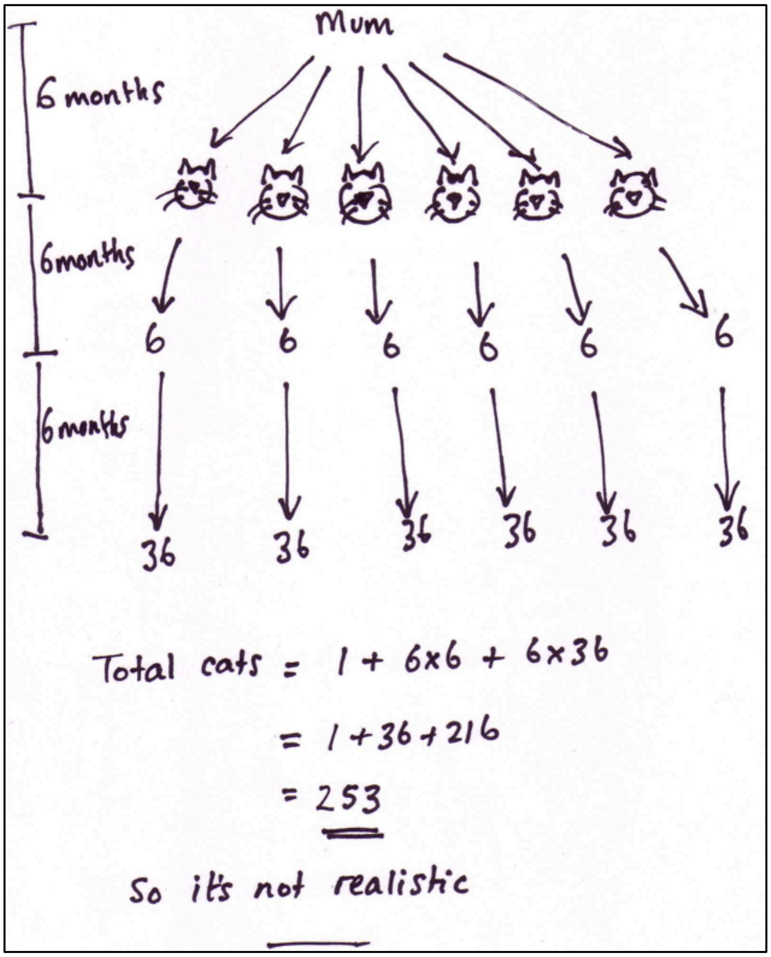
a cat could have 24 kittens
2000 is not realistic

What has Alice done correctly?

What assumptions has she made?

How could this solution be improved?

Sample Responses to Discuss: Wayne



Total cats = $1 + 6 \times 6 + 6 \times 36$

$= 1 + 36 + 216$

$= \underline{\underline{253}}$

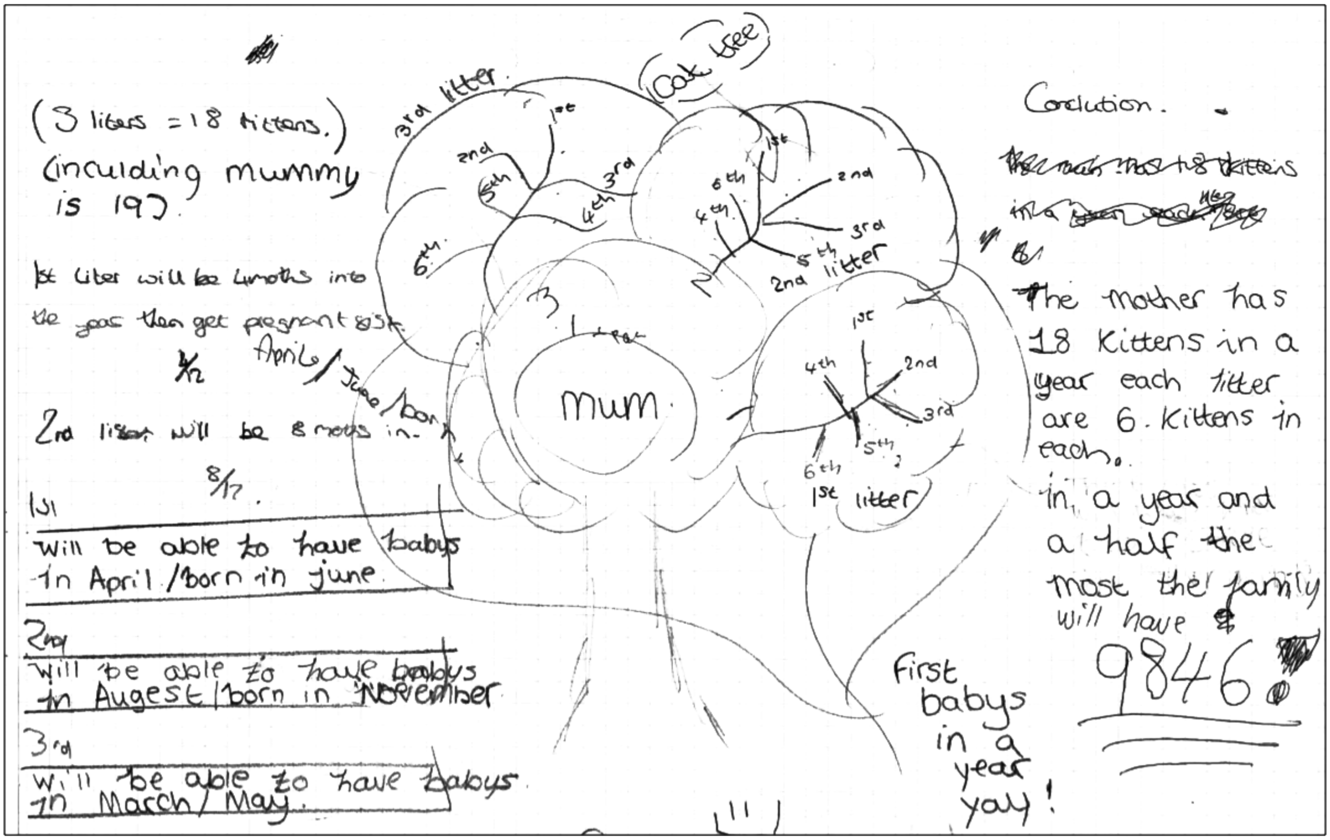
So it's not realistic

What has Wayne done correctly?

What assumptions has he made?

How can Wayne's work be improved?

Sample Responses to Discuss: Ben



(3 liters = 18 kittens.)
(incuilding mummy is 19)

1st liter will be 4moths into the year then get pregnant 8st 4/12

April / June born

2nd litter will be 8 moths in 8/12.

1st
will be able to have babys in April / born in june

2nd
will be able to have babys in Augest / born in November

3rd
will be able to have babys in March / May.

Conclution.

~~the mum has 18 kittens~~ in a ~~[illegible]~~

The mother has 18 Kittens in a year each litter are 6. Kittens in each.
in a year and a half the most the family will have 9846!

First babys in a year yay!

What has Ben done correctly?

What assumptions has he made?

How can Ben's work be improved?

How Did You Work?

Check (√) the boxes and complete the sentences that apply to your work.

1. Check (√) the facts you used:

Length of pregnancy
About
2 months ☐

Age at which a female cat can first get pregnant
About
4 months ☐

Number of kittens in a litter
Usually
4 to 6 ☐

Average number of litters a female cat can have in one year
3 ☐

Age at which a female cat no longer has kittens
About
10 years ☐

2. Our group work was better than my own work ☐

Our joint solution was better because

3. We made some assumptions ☐

These assumptions were

4. Another group has understood my poster ☐

Having Kittens

Cats can't add but

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4 to 6

Average number of litters a female cat can have in one year

3

Age at which a female cat no longer has kittens

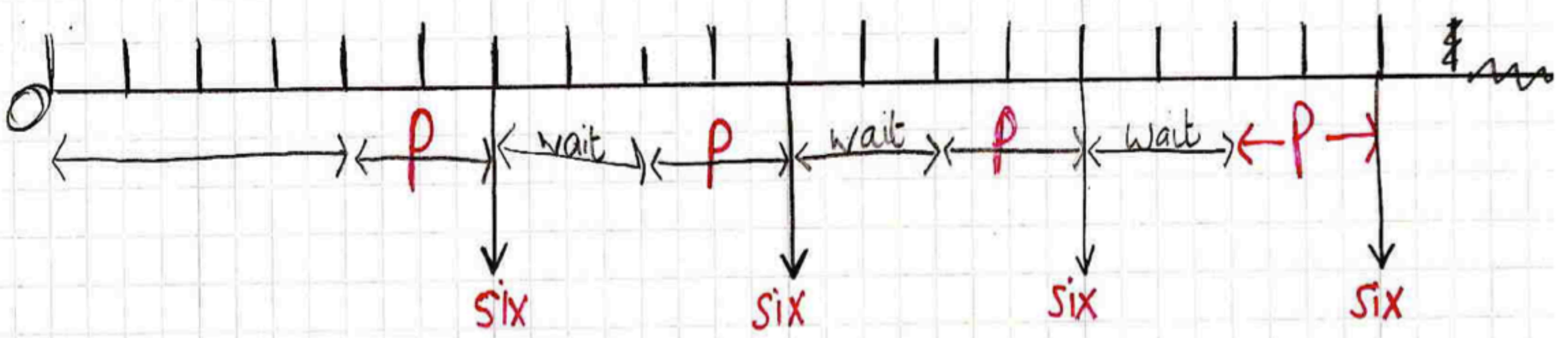
About
10 years

Assessing Sample Student Responses

Your task is to correct the work and write comments about its accuracy and organization.

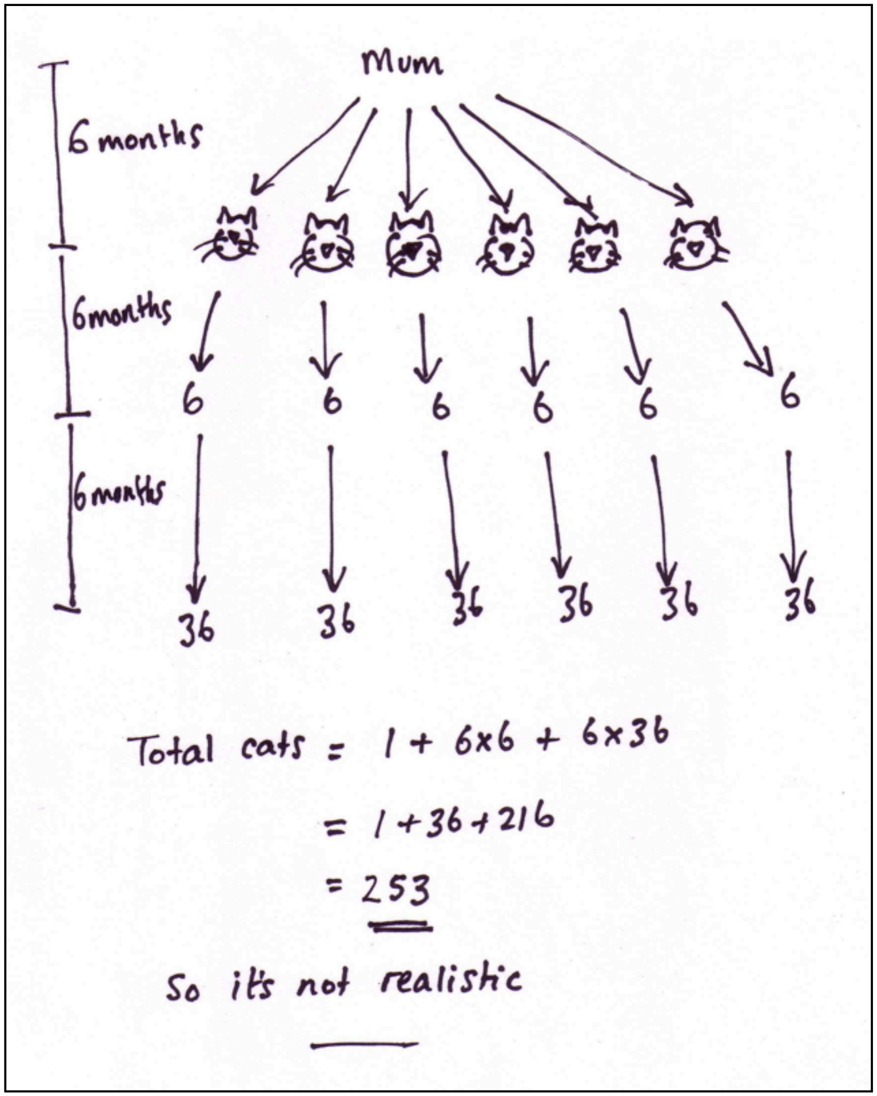
- What has the student done correctly?
- What assumptions has he or she made?
- How could the solution be improved?

Sample Response: Alice



a cat could have 24 kittens
2000 is not realistic

Sample Response: Wayne



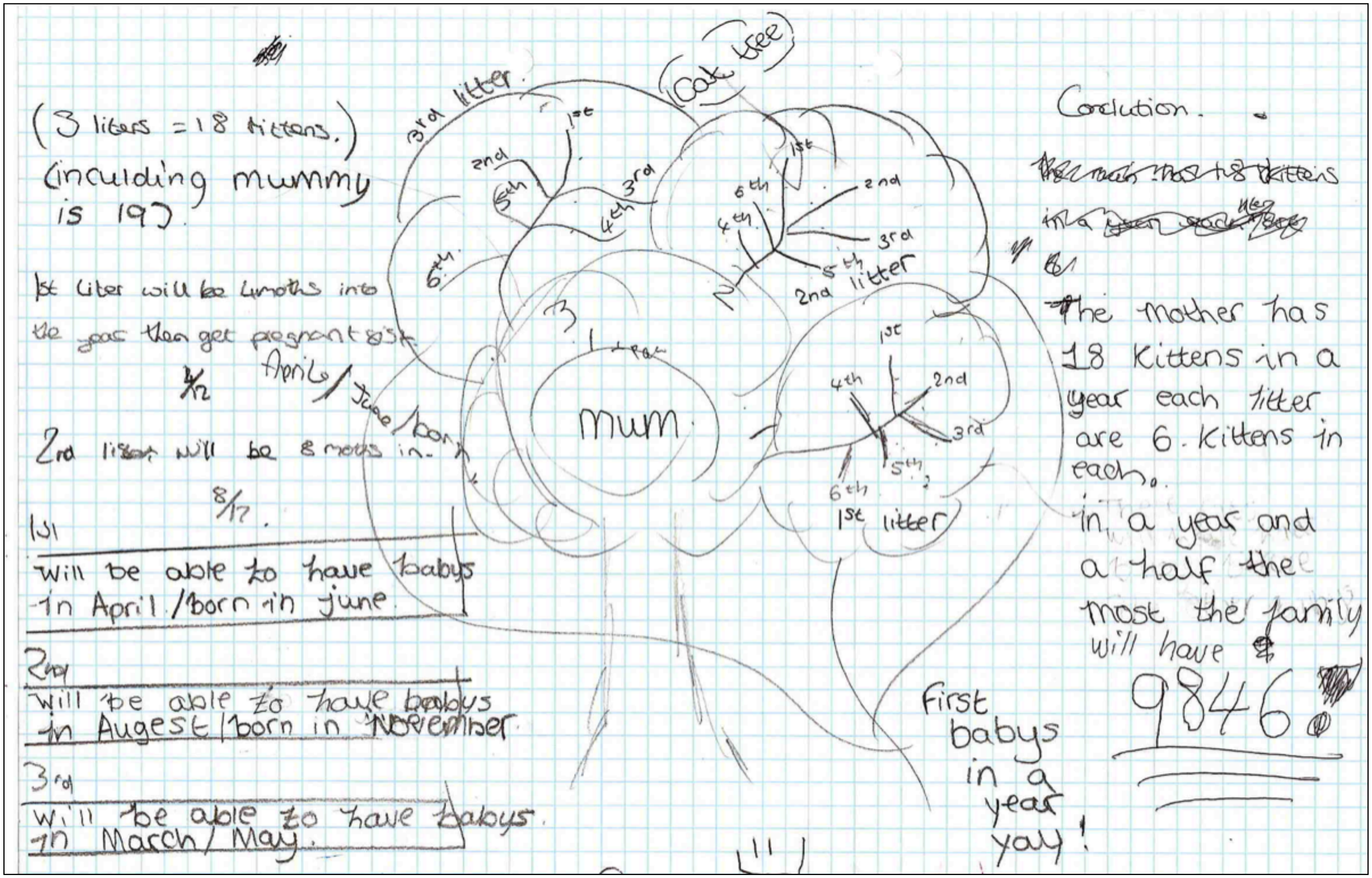
Total cats = $1 + 6 \times 6 + 6 \times 36$

$= 1 + 36 + 216$

$= \underline{\underline{253}}$

So it's not realistic

Sample Response: Ben



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First babys in a year yay!

Conclution.

The mother has 18 Kittens in a year each litter are 6. Kittens in each.
in a year and a half the most the family will have 9846.

Reviewing Work

- I have selected the important facts and used them to solve the problem.
- I am aware of the assumptions I have made and the effect these assumptions have on the result.
- I have used more than one method
- I have checked whether my results make sense and improved my method if need be.
- I have presented my results in a way that will make sense to others.

Mathematics Assessment Project
CLASSROOM CHALLENGES

This lesson was designed and developed by the
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at the
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Malcolm Swan, Nichola Clarke, Clare Dawson, Sheila Evans
with
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It was refined on the basis of reports from teams of observers led by
David Foster, Mary Bouck, and Diane Schaefer
based on their observation of trials in US classrooms
along with comments from teachers and other users.

The central task in this lesson was originally designed by Acumina Ltd. (http://www.acumina.co.uk/) for Bowland Maths (http://www.bowlandmaths.org.uk) and appears courtesy of the Bowland Charitable Trust.

This project was conceived and directed for
MARS: Mathematics Assessment Resource Service
by
Alan Schoenfeld, Hugh Burkhardt, Daniel Pead, and Malcolm Swan
and based at the University of California, Berkeley

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