Lesson 1.1.5 How can I rewrite it?

Investigating Number Patterns

Lesson Objective: Students will investigate number patterns including patterns created by decimals and fractions. Students will be introduced to the vocabulary "terminating" and "repeating" decimal.

Mathematical Practices: Today's lesson is the first example of **looking for and expressing regularity in repeated reasoning**. Encourage students to use this reasoning to **make sense of their problems**.

Length of Activity: One day (approximately 45 minutes)

Core Problems: Problems 1-41 through problem 1-44 part (a)

Materials: Scientific calculators, one per student

Computer with Internet access and projector OR Lesson 1.1.5A Resource Pages (3 in all), copied onto transparencies

Lesson 1.1.5B Resource Page ("Team Roles"), one copy for display (optional)

Technology Notes: This lesson includes a QuickTime™ animation to demonstrate how 0.999... can geometrically be represented as a sum of fractions that converge to 1, available at www.cpm.org/technology. It is recommended that you project this animation at the specified point in problem 1-42 to provoke discussion about using geometric representations of numbers to help make sense of their size. As always, it is encouraged that you test the technology setup before students arrive to be sure that it is working properly. The animation is less than a minute long and moves quickly. You might want to have the students first watch the animation at full speed. Then, replay the animation, pushing the pause button each time the shape is further divided to give students a chance to look more closely at the graphics. Pausing periodically also provides an opportunity to check students' understanding of the divisions that have been made to that point.

If technology is not available, the Lesson 1.1.5A Resource Pages can be reproduced on transparencies and used to produce a similar result. See the notes in the "Suggested Lesson Activity."

Lesson Overview: The goal of this lesson is for students to identify interesting number patterns and recognize that the patterns appear in different ways in different representations. Students use scientific calculators to evaluate expressions quickly in order to focus on patterns and to quickly check predictions. It will be important that each student have access to a calculator in class today.

Suggested Lesson Read the lesson introduction out loud to the class. Emphasize that

Activity: students should use the focus questions as they discuss in their study teams. Students should also be aware that throughout the lesson they should be working on explaining their reasoning and justifying their answers. Students may find connections between steps in a pattern, or between different representations of a number, for example. As they work through the problems, they should focus on finding ways to justify when two numbers or representations are equal.

Problem 1-40 is intended to be a quick introduction to patterns to prepare students for later patterns they will see in the course. Students look for patterns and make predictions in a series of expressions that generate the digits 1 through 9 in reverse order. The expressions in the problem use the symbol "·" to represent multiplication, which may be an unfamiliar symbol for some students. In this course, students will transition to using "*x*" as a variable, so from the outset this course will avoid using an "*x*" to mean multiplication. Take a moment to clarify what the symbol means so that it does not become a barrier to students as they find patterns.

There are many possible patterns that students might try to describe. They should see that the answers are counting backwards from 9 and that the number of digits in the answer is the same as the number that is added to the product of the first two numbers. Another pattern is in the factors of the first two numbers, with the first factor increasing by one digit and the other factor as 8. As you circulate through the classroom, ask students to list the patterns that they see as well as write the next three expressions and their solutions.

Bring the class together for a short discussion when teams have finished. This could be done using a study team strategy called a Whiparound.

List the different patterns students share on a poster, the overhead, or document camera. Leave the list posted for students to refer to as they continue to find patterns in other problems. The list may give students ideas about where to look for patterns as well as language for describing them.

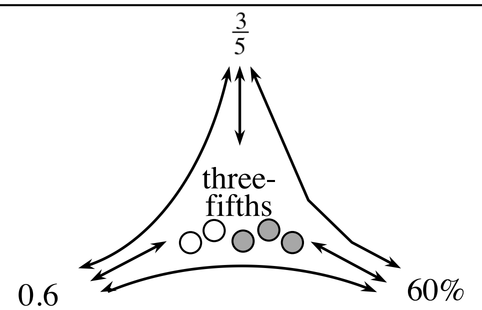
Patterns of Ninths: In problem 1-41, students investigate and make predictions for the patterns that are created when different numbers of ninths are written as decimals. When students make predictions about the decimal equivalent of $\frac{9}{9}$, expect many students to write 0.9999… based on the pattern, without using their knowledge that $\frac{9}{9}=1$ (or that $9 \div 9 = 1$). This is intended to happen, so that students will be surprised when the calculator shows something different.

When teams have had time to discuss the patterns and to formulate theories and justifications about whether 0.999... is equal to 1 (problem 1-42), lead a class discussion for students to share their ideas. To begin this discussion, ask if students' predictions based on the pattern matched their calculator solutions. You may have students recognize that they already knew the decimal equivalents of $\frac{3}{9}$ and $\frac{6}{9}$ because those fractions can be rewritten as $\frac{1}{3}$ and $\frac{2}{3}$, respectively. Expect some debate about whether 0.999... is equivalent to 1. It is normal at this point for students (and adults, for that matter) to be convinced that 0.999… does not equal 1. This sometimes stems from a lack of understanding about the nature of infinity and, in this case, the effect of having decimals repeat infinitely. Ask questions that focus on the nature of infinite decimals, such as, *"If it is less than 1, how much less?"* so that students can see that there is no possible number that represents the difference. Also, if the decimal instead terminates, such as for 0.9999999999999, then there is a difference between it and 1. Encourage teams to share their reasoning, even if their ideas are not fully formed. Their focus at this point should be finding ways to justify their position about whether the two numbers are equal. See the "Mathematical Background" section for more information.

If you want to include some silent writing in this activity, have the students do a Silent Debate. A Silent Debate is a great way to improve the writing/communication skills of your students. It helps them to realize that it is important to be very clear with what they are saying and also learn how to present logical arguments. This is very similar to an oral debate except that it is *silent.* Students pair up and the teacher assigns one person to take the pro side and the other takes the con side. They do not get to choose. The pro goes first. Each pair has one paper and one pencil between them. A topic is given to the students. In this case "0.9999… does equal 1." The pro writes it down and then makes a statement, *in writing,* in support of the topic. Then the con reads the statement and responds to the statement or makes a comment against the topic. This goes on, back and forth, until everyone has had three or four chances to respond.

Geometric Representation: After some discussion, even if students are not yet convinced that the two quantities are equal, suggest that the class consider a visual representation of the portion of 0.999.... This lesson includes a short QuickTime animation demonstration available at www.cpm.org/technology of how each of the decimals 0.9, 0.99, 0.999, etc. can be represented geometrically as the sum of fractions. The diagram then makes it clear how the infinite repeating decimal 0.999..., represented by an infinite sum of fractions, is another representation of the number 1. If you do not have access to a computer with projection technology, the same process can be demonstrated using the Lesson 1.1.5A Resource Pages, copied onto transparencies. During this discussion, show students that the portion 0.9 can be represented by a square divided into 10 equal portions, with nine of those portions shaded (also shown on the Lesson 1.1.5A Resource Page, Diagram A). Then, ask students how they would represent the next digit, or 0.09. Remind students that 0.09 is "nine-hundredths," and so can be represented by dividing the same square (whole) into 100 pieces and shading 9 of them (Lesson 1.1.5A Resource Page, Diagram B). Layer the transparency showing 0.09 on top of the transparency with 0.9 shaded, and continue to layer on additional transparencies of 0.009, and so on in sequence until the whole over time appears to be completely shaded. Discuss how this process could continue with smaller and smaller shaded regions for each 9 that is added to the number to the right of the decimal place. This geometric representation can help to justify to students why the two expressions, $0.\overline{9}$ and 1, represent the same number.

Keep the class together to read problem 1-43, which introduces the term *terminating decimal*. Check that students recognize the difference by giving teams a limited time (2 to 4 minutes) to identify the different numbers as terminating or repeating decimals. Have student volunteers share out answers as a whole class.



Representations of a Portion

Problem 1-44 introduces the portions web that will be used throughout the text as a way to consider four different representations of numbers. These representations include fractions, decimals, percents, and descriptions, which may be geometric or in words. The web is formalized in the Math Notes box at the end of the lesson. Using the fraction $\frac{9}{9}$ as an example, discuss the different pieces of the web before asking students to complete the problem. Use this problem as a formative assessment, along with the suggested "Homework" problems listed below and in Lesson 1.2.1, to help determine if your students need more work with representing fractions as percents and will need to do the optional Lesson 1.2.4.

See the "Universal Access" section of these lesson notes for suggestions on extending the lesson for students who are ready for an additional challenge.

Closure: (5 minutes)

Turn students' attention to problem 1-44, in which they complete two examples of the portions web for themselves. You could also do a Reciprocal Teaching using the words *terminating* and *repeating*. Reciprocal Teaching is done in pairs. Person A pretends that Person B was absent and explains what a terminating decimal is. Then Person B pretends that Person A was absent and explains what a repeating decimal is.

Universal Access:

Academic Language and Literacy Support: In part (e) of problem 1-40 the vocabulary demands of describing the patterns that students see are likely to challenge English language learners and special education students. Providing students with sentence starters could help support students in explaining their ideas. For example, you could offer a starter such as *"One way I see the pattern growing is...."* In addition, remind students that they can use symbols, colors, and arrows to show their thinking. As a teacher, you may choose to use academic words such as *digits*, *products*, *sums*, and *factors* when talking with students.

Although most students may know the word *repeating* and its use in English is consistent with its meaning in math, *terminating* is a word that will not be well known by students. When students begin work on problem 1-43, discuss the meaning of *terminating* in English and how it is related to its meaning in math with the class. Also, help students to recognize that in mathematics, terminating and repeating are opposites.

Additional Challenge: If students have additional time, as an extension, ask them to create portions webs when given a decimal or a percent as a starting place. Converting these representations into fractions will be more formally reviewed later in the chapter. At this time, such problems can be a useful diagnostic assessment of students' prior experience with representations of portions.

Alternately, challenge students to find other decimals that have 2, 3, or more repeating digits, such as $\frac{1}{11} = 0.\overline{09}$. Students will investigate why some decimals repeat and some terminate in Section 2.1.

Team Strategies: While the Collaborative Learning Expectations were introduced in Lesson 1.1.4, it is very important that you continue to reintroduce team roles daily during this chapter to help students thoroughly understand them. Remember that even if your students have prior experience working in teams, it will take them awhile to learn the responsibilities that team roles place on them. It is also critical today and every day that you emphasize that while specific roles come with certain responsibilities (such as asking team questions or keeping close track of time), *everyone* is responsible for thinking mathematically, asking questions, and sharing ideas.

While discussing their reasoning, some team members may get frustrated when they try to justify their thinking and their team members either do not accept their logic or do not understand their argument. Remind teams to have patience and to spend time listening to each other. A "mathematical discussion" or a "logical argument" is not the same thing as a dispute or a fight. When students present an argument in mathematics, they need to rely on providing reasons for statements to help convince others. Each student should realize that if a team member does not agree with his or her statement, it is not personal; it instead offers an opportunity to help that student craft a better, clearer, and more convincing argument. Remind students to give each other time to present their ideas, and to listen and understand before disagreeing.

Assigning students team roles for this activity can also help to structure positive, productive discussions. If you are using roles, take time before the lesson begins to remind students of their responsibilities. A Team Roles Resource Page is provided that includes specific sentence starters to help students perform their roles in this lesson. The information is reprinted below:

Resource Manager:

- Get supplies for your team, and make sure your team cleans up.
- Call the teacher over for team questions.
  "Does anyone have an idea? Should I ask the teacher?"

Facilitator:

- Get your team started by having someone read the task out loud.
  "*Who wants to read?"*
- Make sure everyone understands what to work on.
  "What question are we trying to answer?"
  "Are we ready to move to the next question?"

Recorder/Reporter:

- When your team is called on, share your team's ideas and reasons with the class.
- Make sure that each team member can see the work the team is discussing.
  "Let's work on scratch paper in the middle of the table (or desks) to find the pattern."
  "Can you move your paper to the center so we can all see what you are describing?"
- Make sure each member of your team is able to share their ideas.
  "What other patterns do people see?"
  "Does anyone have other ideas to share?"

Task Manager:

- Make sure no one talks outside your team.
- Help keep your team on task and talking about math.
  "OK, let's focus on the question here."
  "What is the next fraction we need to rewrite?"
- Listen for statements and reasons.
  "Can you explain how you see that?"
  "How do you know those are equal?"

Today is an opportunity to highlight the Task Manager role. The Task Manager makes sure that the team is on task, there is no talking outside the team and that the team is finished in a timely manner.

Remind students of the importance of having one team member's work or another visual focus in the center of the team workspace to make communication easier. The Recorder/Reporter makes sure that each person's ideas are included in the team conversation.

Mathematical

Problem 1-40: This pattern continues through the expression $123456789 \cdot 8 + 9$, but breaks down at the next expression in the

Background: sequence.

Comparing 0.999... and 1: Probably the idea that will be most unusual to students in this lesson is the notion that some numbers can have more than one decimal representation. By this point in their math careers, most students should be comfortable with the idea that $\frac{1}{2}$ and 0.5 are two different representations of the same number—one as a fraction and the other as a decimal. They will probably be surprised, however, to find out that $0.\overline{9}$ and 1.0 also represent the same number. In fact, any terminating decimal will have two different representations. The decimal 0.5 above is really shorthand for 0.500000… and thus can also be written as 0.4999… or $0.4\overline{9}$. Terminating decimals are the only numbers with more than one decimal representation; for example, $\frac{1}{3}$ has only one decimal representation: $0.\overline{3}$.

Most likely, this is your students' first introduction to the idea of an infinite series where an infinite number of numbers are added up (such as $0.9+0.09+0.009+...$) to get a finite number. Of course, there are many infinite series that do not add up to a finite number, but have an infinite sum instead. A famous example is the Harmonic Series ($\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+...$), which has an infinite sum, but gets big very slowly. For example, if the first million terms are added, the sum is less than 14. However, the sum of the entire series still becomes infinite!

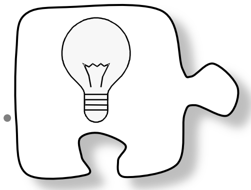
Homework: Problems 1-45 through 1-49

Note: Problems 1-46 and 1-47 are provided to give you material to formatively assess students' understanding of fractions and percents in preparation for optional Lesson 1.2.4. Be sure to assign these problems and look at student work in advance of that lesson.

Students will need access to a calculator for problem 1-45 in the homework. If students do not have access to calculators at home, you may choose to have them complete the problem in class.

Notes to Self:

1.1.5 How can I rewrite it?



Investigating Number Patterns

In the past, you may have looked at number patterns to answer questions or to find the next part of a sequence. Have you ever taken a moment to consider how amazing the mathematics is in those patterns? Today you will investigate number patterns and learn about equivalent ways to write the same number. As a team, you will work to justify why two representations are the same and share your reasoning. In this course, you will often be asked to explain your reasons and justify your answers. When you are reasoning and justifying, you will focus on what makes a statement convincing or how you can explain your ideas. As you look at patterns today, ask your team these questions to help guide your conversations:

What can we predict about the next number in the pattern?

How can we justify our answer?

1-40. In this course, you will transition to using "x" as a variable, so this course will avoid using "x" as a multiplication sign. Instead, the symbol "$\cdot$" will be used to represent multiplication. Use a calculator to calculate the value of each expression below.

a. $1 \cdot 8 + 1$ **[9]**

b. $12 \cdot 8 + 2$ **[98]**

c. $123 \cdot 8 + 3$ **[987]**

d. $1234 \cdot 8 + 4$ **[9876]**

e. What patterns do you see in parts (a) through (d) above? Discuss the patterns with your team. Be sure that when your team agrees on something, it is recorded on each person's paper. **[Sample response: The first factor gains a digit, the added number increases by 1, the answer gets one digit longer and digits are going down starting from 9.]**

f. Use the patterns you found to *predict* the next three expressions and their values. Do not calculate the answers yet. Instead, what do you *think* they will be?
[$12345 \cdot 8 + 5 = 98765$, $123456 \cdot 8 + 6 = 987654$, $1234567 \cdot 8 + 7 = 9876543$]

g. Use your calculator to check the solution for each expression you wrote in part (f). Were your predictions correct? If not, look at the pattern again and figure out how it is changing. **[Answers vary.]**

1-41. Sometimes patterns are not created with addition and multiplication, but with the numbers themselves. For example, when the fractions in the sequence below are changed to decimals, an interesting pattern develops.

$$\frac{1}{9}, \frac{2}{9}, \frac{3}{9}, \text{ and } \frac{4}{9}$$

a. Use your calculator to change each of the fractions above to a decimal. Write each fraction and its equivalent decimal on your paper. **[0.1111…, 0.2222…, 0.3333…, 0.4444…]**

b. Decimals like 0.3333... and the others you found in part (a) are called **repeating decimals** because the digits continue infinitely. Instead of using "..." to show that the numbers repeat, mathematicians write a bar over the digits that repeat, like this: $0.\overline{3}$. It is standard to write the repeating digits just once. For example, $0.2222... = 0.\overline{2}$.

List the next five fractions in the sequence $\frac{1}{9}, \frac{2}{9}, \frac{3}{9}$, and $\frac{4}{9}$. Predict how they will look if they are rewritten as decimals. **[$\frac{5}{9}$ = 0.5555… , $\frac{6}{9}$ = 0.6666… , $\frac{7}{9}$ = 0.7777… , $\frac{8}{9}$ = 0.8888… , $\frac{9}{9}$ = 0.9999... or 1]**

c. Find the decimal equivalents of the five fractions you wrote in part (b) using your calculator. Do they match your predictions? Are there any that are different or that do not follow the pattern? **[Answers vary. Students should note that $\frac{9}{9} \neq 0.\overline{9}$.]**

1-42. Are $0.999...$, $0.\overline{9}$, and 1 equal? How do you know? Discuss this with the class and justify your response. Help others understand what you mean as you explain your thinking. A visual demonstration is available at www.cpm.org/students/technology. **[See the "Suggested Lesson Activity" for possible student responses; during a follow-up discussion, use the slide show from www.cpm.org/technology or the Lesson 1.1.5A Resource Pages to facilitate the discussion.]**

1-43. Decimal numbers that have only a finite number of digits such as 2.173 and 0.04 are called **terminating decimals**. Some fractions can be written as terminating decimals, such as the examples below.

$$\frac{1}{2} = 0.5 \qquad \frac{3}{4} = 0.75$$

Do the decimal equivalents of the numbers below terminate or repeat? Be ready to justify your answer.

a. 0.125
[terminating]

b. $0.\overline{6}$
[repeating]

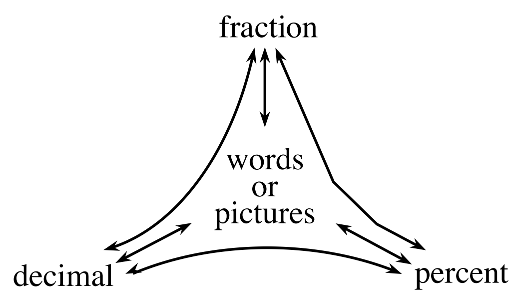
c. $\frac{5}{6}$
[repeating]

d. 4
[terminating]

e. $\frac{2}{5}$
[terminating]

f. − 0.33
[terminating]

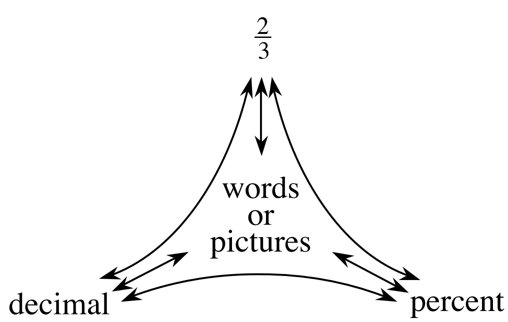
1-44. Representing numbers in multiple ways can help to show what those numbers mean. In problem 1-41, you saw that the fraction $\frac{9}{9}$ (or 1) can be represented as the decimal $0.\overline{9}$, and it can also be represented geometrically with a diagram. Portions can also be represented in words, such as “nine-ninths,” and as **percents**, which are portions of 100. The diagram at right is called a “portions web.”



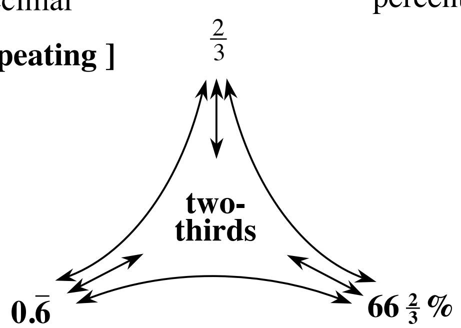
Representations of a Portion

Draw each of the portions webs below on your paper and complete them for the given fractions. In each part, determine if the decimal representation is terminating or repeating.

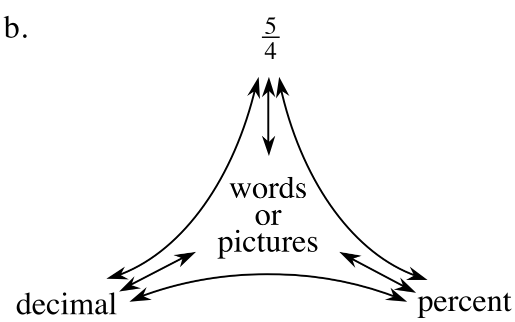
a.



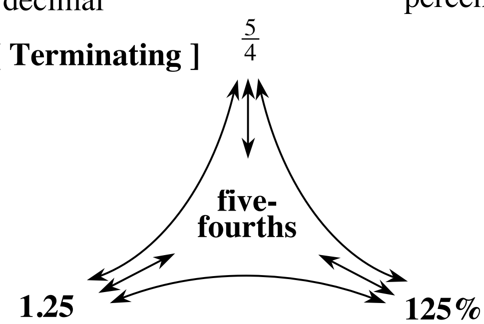
[Repeating]



b.



[Terminating]

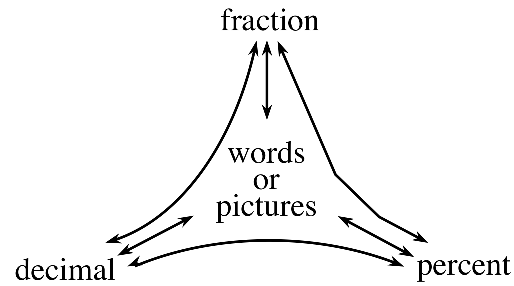


MATH NOTES

METHODS AND MEANINGS

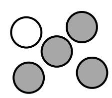
Representations of Portions

The portions web diagram at right illustrates that fractions, decimals, and percents are different ways to represent a portion of a number. Portions can also be represented in words, such as "four-fifths" or "seven-fourths," or with diagrams such as those shown below. A complete portions web is shown below right.



Representations of a Portion

$\frac{4}{5}$ of the objects are shaded:



150% of one circle is shaded: