General Certificate of Education
Advanced Level Examination
June 2010

Use of Mathematics (Pilot) USE3/PM

Mathematical Comprehension

Preliminary Material

Data Sheet

To be opened and issued to candidates between Thursday 6 May 2010 and Thursday 20 May 2010

REMINDER TO CANDIDATES

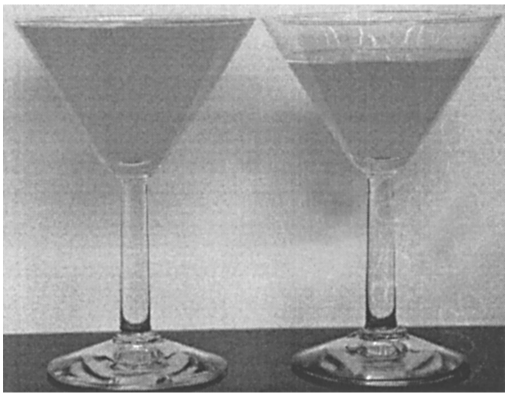
YOU MUST **NOT** BRING THIS DATA SHEET
WITH YOU WHEN YOU SIT THE EXAMINATION.
A CLEAN COPY WILL BE MADE AVAILABLE.

Half full or half empty?

Johannes Kepler is a well-known 17th century astronomer. He was particularly influential because of the importance of his laws of planetary motion. However, he was also an eminent mathematician. His dissatisfaction at how wine merchants used rough and ready methods to estimate the volume of wine in the barrels they delivered led him to work out mathematical methods that were more accurate. These methods involved finding volumes of revolution, and might be considered as forerunners to calculus.

Methods similar to those devised by Kepler allow you to calculate the depth to which different glasses can be filled so that each contains exactly the same amount of liquid. Although, perhaps, the easiest way to do this is to measure out the same amount of liquid into each, the mathematics involved is interesting.

Figure 1 *Two identical glasses with one containing twice the volume of liquid of the other*



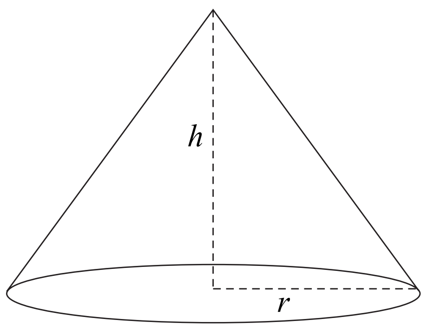
For example, consider the picture of two identical cocktail glasses in **Figure 1**. You might be surprised to learn that one contains exactly half the amount of liquid of the other.

How can one calculate the depth to which a glass should be filled so that it is half full?
In the case of the glasses shown in **Figure 1**, this is not too difficult as we could assume that the glass is conical and use a formula for calculating its volume (see **Figure 2**).

For the particular glass shown in **Figure 1**, the radius of the maximum circular cross-section is 4.6 cm, and the depth of the glass is 5.5 cm. The maximum volume of liquid, $V\,\text{cm}^3$, that can be poured into the glass is approximately $122\,\text{cm}^3$.

The depth to which to fill the glass so that it is half full can be found to be 4.37 cm.

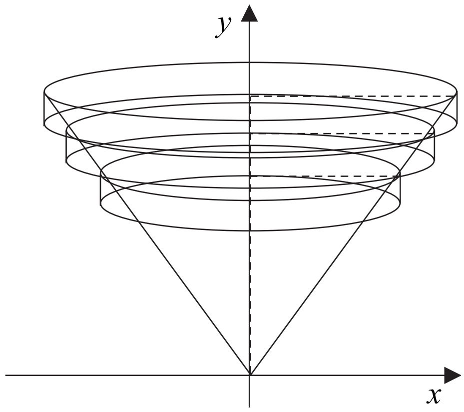
Figure 2 *Formula for the volume of a cone*



volume of a cone, $V = \frac{1}{3}\pi r^2 h$

An alternative method of calculating the volume of the conical glass is to assume that it is made of a series of concentric cylinders. **Figure 3** shows just three of a series of these, together with x and y axes aligned horizontally and vertically respectively, with the vertex of the cone as the origin. If eleven such cylinders were used to make the entire conical glass, each would have a depth of 0.5 cm.

Figure 3 *Modelling a cone with a series of concentric cylinders*



The table in **Figure 4** gives, for each successive cylinder, its radius, x cm, and the vertical distance, y cm, of its upper circular face from the origin.

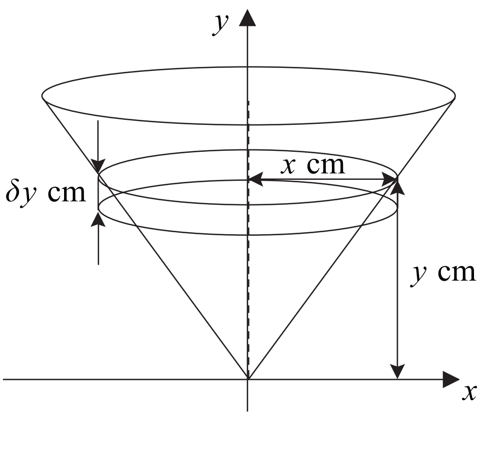
Figure 4 *Table showing dimensions of eleven cylinders used to model the conical cocktail glass*

Cylinder no	y (cm)	x (cm)	Volume (cm^3)
1	5.50	4.60	33.24
2	5.00	4.18	27.45
3	4.50	3.76	22.21
4	4.00	3.35	17.63
5	3.50	2.93	13.49
6	3.00	2.51	9.90
7	2.50	2.09	6.86
8	2.00	1.67	4.38
9	1.50	1.25	2.45
10	1.00	0.84	1.11
11	0.50	0.42	0.28
		TOTAL	139.00

This gives an over-estimate of the volume of the entire cone, but using more cylinders will allow you to calculate a more accurate estimate. The radius, x cm, of a cylinder y cm from the origin can be found using $y = \frac{5.5}{4.6}x$. Therefore, the volume, δV, of a cylinder with this radius and thickness δy (see **Figure 5**) would be given by

$$\delta V = \pi x^2 \times \delta y = \pi \left(\frac{4.6}{5.5} y\right)^2 \delta y$$

Figure 5 *Dimensions of a general cylinder used to build a model of a conical cocktail glass*



Summing all cylinders to give the total volume of the cone gives $V = \sum \pi x^2 \delta y$.

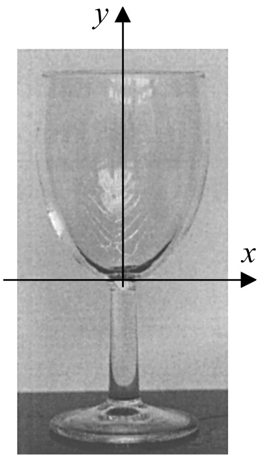
In the limit as $\delta y \to 0$, the total volume of the cone is, therefore, given by

$$V = \int_0^{5.5} \pi \left(\frac{4.6}{5.5} y\right)^2 \mathrm{d}y$$

leading to the result $V = 121.87\,\text{cm}^3$.

Such techniques can be used to find the volumes of other glasses if you know, or can find, a function to describe the vertical cross-section of the glass. For example, consider the wine glass shown in **Figure 6**. This also gives data for x and y coordinates, referring to axes as shown, which were taken by measuring the glass.

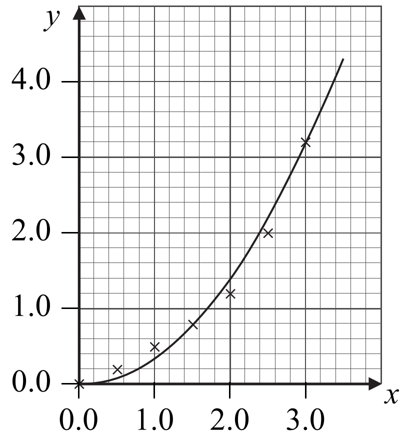
Figure 6 *Wine glass and data*



x	y
0.0	0.0
0.5	0.2
1.0	0.5
1.5	0.8
2.0	1.2
2.5	2.0
3.0	3.2
3.5	5.7

In this case, it is not easy to find a simple function that is a good match to the outline of the glass. However, as its graph, plotted in **Figure 7** with the data, shows, the function $y = 0.35x^2$ might be considered a good enough approximation.

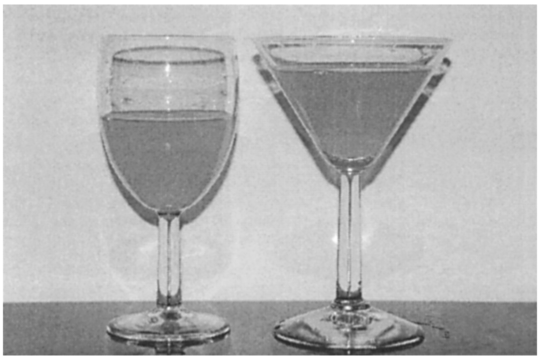
Figure 7 *Modelling the outline of a wine glass*



In this case, the volume of liquid in the glass when filled to a depth d cm, is given by $V = \int_0^d \pi x^2 \mathrm{d}y$. Using the function $y = 0.35x^2$ to model the wine glass gives the volume of the liquid when the glass is filled to a depth of d cm as $V = 4.49d^2 \text{ cm}^3$.

Hence, to fill this glass so that it has the same amount of liquid as the half-full cocktail glass of **Figure 1**, you can now find that it should be filled to a depth of 3.69 cm. The two glasses containing this same amount of liquid are shown in **Figure 8**.

Figure 8 *Two glasses containing the same volume of liquid*



As the photographs in **Figures 1 and 8** demonstrate, it is perhaps not surprising that the wine merchants who delivered wine to Johannes Kepler were prone to making errors in their estimates!

END OF DATA SHEET